This exam will consist of two parts and each part will have 3 multipart questions. Each of the 6 questions is worth 50 points for a total of 300 points. The two part of the exam are (I) Linear Least Squares and (II) Quadratic Optimization. In each part, the first question concerns definitions, theorems, and proofs, and the remaining two questions are computational. This format is very much like the quizzes. More detailed descriptions of the questions are given below. This is then followed by a list of sample questions.

I Linear Least Squares

1 Theory Question: For this question you will need to review all of the vocabulary words as well as the theorems for Linear Least Squares. You will also be asked to establish properties of the linear algebraic, structures such as orthogonal projections, and the solution set.

2 Linear Algebra: Here you may be asked to compute the solution to an associated linear system, such as the normal equations, compute an LU and/or QR factorization, compute an orthogonal projection onto a subspace, and /or compute a representation for the solution for a specially structured linear least squares problem.

3 Other Computations: Here you may be asked to compute the solution set to a linear least squares problem, compute and orthogonal projection onto a subspace.

II Quadratic Optimization

4 Theory Question: For this question you will need to review all of the vocabulary words as well as the theorems in Chapter 4 for quadratic optimization problems. You will also be asked to establish properties of the linear algebraic structures and the solution set.

5 Linear Algebra: Here you may be asked to compute an eigenvalue decomposition and/or a Cholesky factorization, and/or a representation for the solution for a specially structured quadratic optimization problem.

6 Other Computations: Here you may be asked to compute the solution set to a quadratic optimization problem possibly with constraints.

# Sample Questions

(I) Linear Least Squares

**Question** 1:
Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, and consider the linear least squares problem

$$\mathcal{LLS} \qquad \min \frac{1}{2}\|Ax-b\|_2^2\ .$$

a. Show that the matrix $A^TA$ is always positive semi-definite and provide necessary and sufficient condition on $A$ under which $A^TA$ is positive definite.

b. Show that $\mathrm{Nul}(A^TA) = \mathrm{Nul}(A)$ and $\mathrm{Ran}(A^TA) = \mathrm{Ran}(A^T)$.

c. Show that $\mathcal{LLS}$ always has a solution.

d. State and prove a necessary and sufficient condition on the matrix $A \in \mathbb{R}^{m\times n}$ under which $\mathcal{LLS}$ has a unique global optimal solution.

e. Decribe the QR factorization of $A$ and show how it can be used to construct a solution to $\mathcal{LLS}$.

f. If $\mathrm{Nul}(A) = \{0\}$, show that $(A^TA)^{-1}$ is well defined and that $P = A(A^TA)^{-1}A^T$ is the orthogonal projection onto $\mathrm{Ran}(A)$ and that

$$\frac{1}{2}\|(I-P)b\|_2^2 = \min \frac{1}{2}\|Ax-b\|_2^2 \ .$$

g. Let $A \in \mathbb{R}^{m\times n}$ be such that $\mathrm{Ran}(A) = \mathbb{R}^m$ and set $P = A^T(AA^T)^{-1}A$. Show that $Px^0 = A^T(AA^T)^{-1}b$ for every $x^0 \in \mathbb{R}^n$ satisfying $Ax^0 = b$ and that $\hat{x} := A^T(AA^T)^{-1}b$ is the unique solution to the problem

$$\min \frac{1}{2}\|x\|_2^2 \quad \text{subject to} \quad Ax = b \ .$$

**Question** 2:

(A) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

a. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.
b. Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.
c. Compute the QR factorization of $A$.
d. Compute the LU factorization of $A^T$.

(B) Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

a. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.
b. Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.
c. Compute the QR factorization of $A$.
d. Compute the LU factorization of $A^T$.

**Question** 3:

A Let $a \in \mathbb{R}$ and consider the function

$$f(x_1, x_2, x_3) = \frac{1}{2}[(2x_1 - 2a^4)^2 + (x_1 - x_2)^2 + (ax_2 + x_3)^2].$$

(a) Write this function in the form of the objective function for a linear least squares problem by specifying the matrix $A$ and the vector $b$.
(b) Describe the solution set of this linear least squares problem as a function of $a$.

(B) Find the quadratic polynomial $p(t) = x_0 + x_1t + x_2t^2$ that best fits the following data in the least-squares sense:

| $t$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $2$ | $-10$ | $0$ | $2$ | $1$ |

(C) Consider the problem LLS with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) What are the normal equations for this $A$ and $b$.

(b) Solve the normal equations to obtain a solution to the problem LLS for this $A$ and $b$.

(c) What is the general reduced QR factorization for this matrix $A$?

(d) Compute the orthogonal projection onto the range of $A$.

(e) Use the recipe

$$\begin{aligned} AP &= Q[R_1\ R_2] && \text{the general reduced QR factorization} \\ \hat{b} &= Q^T b && \text{a matrix-vector product} \\ \bar{w}_1 &= R_1^{-1}\hat{b} && \text{a back solve} \\ \bar{x} &= P\begin{bmatrix} R_1^{-1}\hat{b} \\ 0 \end{bmatrix} && \text{a matrix-vector product.} \end{aligned}$$

to solve LLS for this $A$ and $b$.

(f) If $\bar{x}$ solves LLS for this $A$ and $b$, what is $A\bar{x} - b$?

(II) Quadratic Optimization

**Question** 4:

Consider the function

$$f(x) = \frac{1}{2}x^T Q x + c^T x,$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric and $c \in \mathbb{R}^m$.

1. What is the eigenvalue decomposition of $Q$?

2. Give necessary and sufficient conditions on $Q$ and $c$ for which there exists a solution to the problem $\min_{x\in\mathbb{R}^n} f(x)$. Justify your answer.

3. If $Q$ is positive definite, show that there is a nonsingular matrix $L$ such that $Q = LL^T$.

4. With $Q$ and $L$ as defined in the part (b), show that

$$f(x) = \frac{1}{2}\|L^T x + L^{-1}c\|_2^2 - \frac{1}{2}c^T Q^{-1} c.$$

5. If $f$ is convex, under what conditions is $\min_{x\in\mathbb{R}^n} f(x) = -\infty$?

6. Let $\hat{x} \in \mathbb{R}^n$ and $S$ be a subspace of $\mathbb{R}^n$. Give necessary and sufficient conditions on $Q$ and $c$ for which there exists a solution to the problem

$$\min_{x\in\hat{x}+S} f(x)\ .$$

7. Show that every local solution to the problem $\min_{x\in\mathbb{R}^n} f(x)$ is necessarily a global solution.

**Question** 5:

(A) Compute the Cholesky factorizations of the following matrices.

$$(a)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (b)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$(c)\ H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad (d)\ H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} .$$

(B) Compute the eigenvalue decomposition of the following matrices.

$$(a)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (b)\ H = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$(c)\ H = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad (d)\ H = \begin{bmatrix} 5 & -1 & -1 & 1 \\ -1 & 4 & 2 & -1 \\ -1 & 2 & 4 & -1 \\ 1 & -1 & -1 & 5 \end{bmatrix}$$

**Question** 6:

(A) For each of the matrices $H$ and vectors $g$ below determine the optimal value in $\mathcal{Q}$. If an optimal solution exists, compute the complete set of optimal solutions.

a.

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} .$$

b.

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} .$$

c.

$$H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} .$$

(B) Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $g \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} .$$

Does there exists a vector $u \in \mathbb{R}^3$ such that $f(tu) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct $u$.

(C) Consider the linearly constrained quadratic optimization problem

$$\mathcal{Q}(H, g, A, b) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Hx + g^T x \\ \text{subject to} & Ax = b \ , \end{array}$$

where $H$, $A$, $g$, and $b$ are given by

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \ g = (1,1,1)^T, \ b = (4,2)^T, \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

a. Compute a basis for the null space of $A$.

b. Solve the problem $\mathcal{Q}(H, g, A, b)$ two different ways. One should use a basis for the null space of $A$ and the other should not.

(D) Let $H \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, $r \in \mathbb{R}^n \setminus \mathrm{Span}[\mathsf{e}]$, $\mu \in \mathbb{R}$. Solve the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x \\ \text{subject to} & \mathsf{e}^T x = 1 \quad \text{and} \quad r^T x = \mu \ , \end{array}$$

where $\mathsf{e} := (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ is the vector of all ones and $\mu \in \mathbb{R}$.