May 9, 2014

MATH 408 MIDTERM EXAM OUTLINE

This exam will consist of two parts and each part will have 3 multipart questions. Each of the 6 questions is worth 50 points for a total of 300 points. The two part of the exam are (I) Linear Least Squares and (II) Quadratic Optimization. In each part, the first question concerns definitions, theorems, and proofs, and the remaining two questions are computational. This format is very much like the quizzes. More detailed descriptions of the questions are given below. This is then followed by a list of sample questions.

I Linear Least Squares

1 Theory Question: For this question you will need to review all of the vocabulary words as well as the theorems for Linear Least Squares. You will also be asked to establish properties of the linear algebraic, structures such as orthogonal projections, and the solution set.

2 Linear Algebra: Here you may be asked to compute the solution to an associated linear system, such as the normal equations, compute an LU and/or QR factorization, compute an orthogonal projection onto a subspace, and /or compute a representation for the solution for a specially structured linear least squares problem.

3 Other Computations: Here you may be asked to compute the solution set to a linear least squares problem, compute and orthogonal projection onto a subspace.

II Quadratic Optimization

4 Theory Question: For this question you will need to review all of the vocabulary words as well as the theorems in Chapter 4 for quadratic optimization problems. You will also be asked to establish properties of the linear algebraic structures and the solution set.

5 Linear Algebra: Here you may be asked to compute an eigenvalue decomposition and/or a Cholesky factorization, and/or a representation for the solution for a specially structured quadratic optimization problem.

6 Other Computations: Here you may be asked to compute the solution set to a quadratic optimization problem possibly with constraints.

Sample Questions

(I) Linear Least Squares

Question 1:

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, and consider the linear least squares problem

$$\mathcal{LLS} \qquad \min \frac{1}{2}\|Ax-b\|_2^2 \ .$$

a. Show that the matrix A^TA is always positive semi-definite and provide necessary and sufficient condition on A under which A^TA is positive definite.

Solution

$x^TA^TAx = \|Ax\|_2^2 \geq 0 \ \forall\, x \in \mathbb{R}^n$ so A^TA is always positive semi-definite. A^TA is positive definite if and only if $0 < x^TA^TAx = \|Ax\|_2^2 \ \forall\, x \in \mathbb{R}^n \setminus \{0\}$, or equivalently, $\text{Nul}(A) = \{0\}$.

b. Show that $\mathrm{Nul}(A^TA) = \mathrm{Nul}(A)$ and $\mathrm{Ran}(A^TA) = \mathrm{Ran}(A^T)$.

Solution

$A^TAx = 0$ implies $x^TA^TAx = 0$, i.e. $\|Ax\|_2^2 = 0$, hence $Ax = 0$. The other direction is trivial.

$$\mathrm{dimRan}(A^TA) + \mathrm{dimNull}(A^TA) = n$$
$$\mathrm{dimRan}(A^T) + \mathrm{dimNull}(A) = n.$$

Since $\mathrm{dimNull}(A) = \mathrm{dimNull}(A^TA)$, hence $\mathrm{dimRan}(A^TA) = \mathrm{dimRan}(A^T)$. Moreover $\mathrm{Ran}(A^TA) \subset \mathrm{Ran}(A^T)$. Therefore $\mathrm{Ran}(A^TA) = \mathrm{Ran}(A^T)$.

c. Show that $\mathcal{LLS}$ always has a solution.

Solution

Theorem 2.2 from Chapter 3.

d. State and prove a necessary and sufficient condition on the matrix $A \in \mathbb{R}^{m\times n}$ under which $\mathcal{LLS}$ has a unique global optimal solution.

Solution

$\mathrm{Nul}(A) = \{0\}$.

e. Decribe the QR factorization of A and show how it can be used to construct a solution to $\mathcal{LLS}$.

Solution

Chaper 3, section 5.2.

f. If $\mathrm{Nul}(A) = \{0\}$, show that $(A^TA)^{-1}$ is well defined and that $P = A(A^TA)^{-1}A^T$ is the orthogonal projection onto $\mathrm{Ran}(A)$ and that

$$\frac{1}{2}\|(I-P)b\|_2^2 = \min \frac{1}{2}\|Ax-b\|_2^2 \ .$$

Solution

By part b), $\mathrm{Nul}(A^TA) = \mathrm{Nul}(A) = \{0\}$, then A^TA is invertible.

Theorem 3.1 from Chapter 3 and the paragraph after it.

g. Let $A \in \mathbb{R}^{m\times n}$ be such that $\mathrm{Ran}(A) = \mathbb{R}^m$ and set $P = A^T(AA^T)^{-1}A$. Show that $Px^0 = A^T(AA^T)^{-1}b$ for every $x^0 \in \mathbb{R}^n$ satisfying $Ax^0 = b$ and that $\hat{x} := A^T(AA^T)^{-1}b$ is the unique solution to the problem

$$\min \frac{1}{2}\|x\|_2^2 \quad \text{subject to} \quad Ax = b \ .$$

Solution

Theorem 4.1 from Chapter 3.

Question 2:

(A) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

a. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.

Solution: *After applying Gram-Schmidt to the columns of A and them writing them as the columns of the matrix Q, we obtain*

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix},$$

This gives the orthogonal projection

$$QQ^T = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}.$$

b. Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.

Solution: *Since* $\mathrm{Null}(A^T) = \mathrm{Ran}(A)^{\perp}$*, the projection onto* $\mathrm{Null}(A^T)$ *is just* $I - QQ^T$*, where* Q *is given above (see the discussion on page 28 of the notes):*

$$I - QQ^T = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

c. Compute the QR factorization of A.

Solution:

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

d. Compute the LU factorization of A^T.

Solution:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA = LU.$$

(B) Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

a. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.

Solution: *Apply Lemma 3.1 on page 29 of the notes to answer this question. This lemma tells us that we need to obtain an orthonormal basis for the range of A, write these basis vectors as the columns of a matrix Q, and then QQ^T is the orthogonal projection onto* $\mathrm{Ran}(A)$.

Begin by applying the Gram-Schmidt orthogonalization process to the columns of A since the range of A is the linear span of its columns. This yields the two vectors

$$q^1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad \textit{and} \quad q^2 = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \quad \textit{so that} \;\; Q = \begin{bmatrix}1/2 & -1/2\\1/2 & 1/2\\1/2 & -1/2\\1/2 & 1/2\end{bmatrix}.$$

Notice that this matrix Q is the same matrix that appears in the QR factorization of A. The orthogonal projection can now be written as

$$QQ^T = \begin{bmatrix}1/2 & 0 & 1/2 & 0\\0 & 1/2 & 0 & 1/2\\1/2 & 0 & 1/2 & 0\\0 & 1/2 & 0 & 1/2\end{bmatrix}.$$

b. Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.

Solution:

$\mathrm{Nul}(A^T) = \mathrm{Ran}(A)^\perp$.

$$P_{\mathrm{Nul}(A^T)} = I - P_{\mathrm{Ran}(A)} = \begin{bmatrix}\frac{1}{2} & 0 & -\frac{1}{2} & 0\\0 & \frac{1}{2} & 0 & -\frac{1}{2}\\-\frac{1}{2} & 0 & \frac{1}{2} & 0\\0 & -\frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}$$

c. Compute the QR factorization of A.

Solution:

$$QR = \begin{bmatrix}1/2 & -1/2\\1/2 & 1/2\\1/2 & -1/2\\1/2 & 1/2\end{bmatrix}\begin{bmatrix}2 & 0 & 2\\0 & 2 & 2\end{bmatrix}.$$

d. Compute the LU factorization of A^T.

Solution:

$$L = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 1 & 1\end{bmatrix}, U = \begin{bmatrix}1 & 1 & 1 & 1\\0 & 2 & 0 & 2\\0 & 0 & 0 & 0\end{bmatrix}$$

Question 3:

A Let $a \in \mathbb{R}$ and consider the function

$$f(x_1, x_2, x_3) = \frac{1}{2}[(2x_1 - 2a^4)^2 + (x_1 - x_2)^2 + (ax_2 + x_3)^2].$$

(a) Write this function in the form of the objective function for a linear least squares problem by specifying the matrix A and the vector b.

Solution:

$$A = \begin{bmatrix}2 & 0 & 0\\1 & -1 & 0\\0 & a & 1\end{bmatrix}, b = \begin{bmatrix}2a^4\\0\\0\end{bmatrix}$$

(b) Describe the solution set of this linear least squares problem as a function of a.

Solution:

$(x_1, x_2, x_3) = (a^4, a^4, -a^5)$.

(B) Find the quadratic polynomial $p(t) = x_0 + x_1 t + x_2 t^2$ that best fits the following data in the least-squares sense:

$$\begin{array}{c|ccccc} t & -2 & -1 & 0 & 1 & 2 \\ \hline y & 2 & -10 & 0 & 2 & 1 \end{array}.$$

Solution: Write it as an LLS problem where

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -10 \\ 0 \\ 2 \\ 1 \end{bmatrix},$$

and solving the LLS gives us $(x_0, x_1, x_2) = (-3, 1, 1)$.

(C) Consider the problem LLS with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) What are the normal equations for this A and b.

Solution: *The normal equations are $A^T Ax = A^T b$ (see Theorem 2.1 on page 26 of the notes), where*

$$A^T A = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix} \quad \textit{and} \quad A^T b = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

(b) Solve the normal equations to obtain a solution to the problem LLS for this A and b.

Solution: *The set of all solutions to the normal equations are*

$$x = \frac{1}{4} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad t \in \mathbb{R}.$$

(c) What is the general reduced QR factorization for this matrix A?

Solution:

$$QR = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

(d) Compute the orthogonal projection onto the range of A.

Solution:

See Question 2(B) (a).

(e) Use the recipe

$$\begin{aligned} AP &= Q[R_1\ R_2] && \text{the general reduced QR factorization} \\ \hat{b} &= Q^T b && \text{a matrix-vector product} \\ \bar{w}_1 &= R_1^{-1}\hat{b} && \text{a back solve} \\ \bar{x} &= P\begin{bmatrix} R_1^{-1}\hat{b} \\ 0 \end{bmatrix} && \text{a matrix-vector product.} \end{aligned}$$

to solve LLS for this A and b.

Solution:

$$\begin{aligned} \hat{b} &= Q^T b = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} \\ \bar{w}_1 &= R_1^{-1}\hat{b} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ -1/4 \end{pmatrix} \\ \bar{x} &= P\begin{bmatrix} R_1^{-1}\hat{b} \\ 0 \end{bmatrix} = \begin{pmatrix} 3/4 \\ -1/4 \\ 0 \end{pmatrix}. \end{aligned}$$

Note that this gives only the particular solution given in Part (b) above and not the entire solution set. How might you recover the entire solution set from th QR factorization?

(f) If $\bar{x}$ solves LLS for this A and b, what is $A\bar{x} - b$?

Solution:

$$A\bar{x} - b = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

(II) Quadratic Optimization

Question 4:

Consider the function

$$f(x) = \frac{1}{2}x^T Q x + c^T x,$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric and $c \in \mathbb{R}^m$.

1. What is the eigenvalue decomposition of Q?

   Solution: Theorem 1.1 from Chapter 4.

2. Give necessary and sufficient conditions on Q and c for which there exists a solution to the problem $\min_{x\in\mathbb{R}^n} f(x)$. Justify your answer.

   Solution: Theorem 2.1 from Chapter 4.

3. If Q is positive definite, show that there is a nonsingular matrix L such that $Q = LL^T$.

   Solution:

   Eigenvalue decomposition of $Q = UDU^T$, where $D = \mathrm{diag}\{d_1, \cdots, d_n\}$ where d_i $i = 1, \cdots, n$ are the eigenvalues of Q. Since Q is positive definite, then $d_i > 0$ $i = 1, \cdots, n$. Set $D^{\frac{1}{2}} = \mathrm{diag}\{d_1^{\frac{1}{2}}, \cdots, d_n^{\frac{1}{2}}\}$ and $L = UD^{\frac{1}{2}}$, then $Q = LL^T$.

4. With Q and L as defined in the part 3, show that

$$f(x) = \frac{1}{2}\|L^T x + L^{-1}c\|_2^2 - \frac{1}{2}c^T Q^{-1} c.$$

Solution:

$$\begin{aligned} f(x) &= \frac{1}{2}x^T Q x + c^T x \\ &= \frac{1}{2}x^T L L^T x + c^T x \\ &= \frac{1}{2}\|L^T x + L^{-1}c\|_2^2 - \frac{1}{2}c^T Q^{-1} c \end{aligned}$$

5. If f is convex, under what conditions is $\min_{x\in\mathbb{R}^n} f(x) = -\infty$?

   Solution: $-c \notin \mathrm{Ran}(Q)$.

6. Let $\hat{x} \in \mathbb{R}^n$ and S be a subspace of $\mathbb{R}^n$. Give necessary and sufficient conditions on Q and c for which there exists a solution to the problem

$$\min_{x\in\hat{x}+S} f(x) \ .$$

   Solution:

   Theorem 3.1 from Chapter 4.

7. Show that every local solution to the problem $\min_{x\in\mathbb{R}^n} f(x)$ is necessarily a global solution.

   Solution:

   Theorem 3.1 from Chapter 4.

Question 5:

(A) Compute the Cholesky factorizations of the following matrices.

$$(a)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (b)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$(c)\ H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad (d)\ H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} .$$

Solution

(a) Positive definite by the determinant test.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

and so

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{4/3} \end{bmatrix} .$$

(b) Obviously not positive definite since it has a -2 on the diagonal. Nonetheless,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & -8/3 \end{bmatrix}$$

and so the last pivot being negative implies that H is not positive semi-definite.

(c) H is positive semi-definite.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2/5 & 1 & 0 \\ 1/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -1 \\ 0 & 1/5 & -3/5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2/5 & -1/5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

and so

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/5 & 1 & 0 \\ -1/5 & -3 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{1/5} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(d)H is positive semi-definite.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

and so

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(B) Compute the eigenvalue decomposition of the following matrices.

$$(a)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (b)\ H = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$(c)\ H = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad (d)\ H = \begin{bmatrix} 5 & -1 & -1 & 1 \\ -1 & 4 & 2 & -1 \\ -1 & 2 & 4 & -1 \\ 1 & -1 & -1 & 5 \end{bmatrix}$$

Solution:

$$H = UDU^T.$$

(a)

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{bmatrix}$$

(b)

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)

$$U = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(d)

$$U = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{\sqrt{2}}{2} & 1 \\ -\frac{\sqrt{2}}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Question 6:

(A) For each of the matrices H and vectors g below determine the optimal value in $\mathcal{Q}$. If an optimal solution exists, compute the complete set of optimal solutions.

a.

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:
The eigenvalues are $2, 2 \pm \sqrt{2}$ so H is positive definite. Therefore the unique optimal solution is given by $-H^{-1}g = (-2, 1, -1)^T$.

b.

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:
The characteristic polynomial is $p(\lambda) = \lambda^3 - 2\lambda^2 - 6\lambda + 8$. Sketching the graph shows one negative and two positive eigenvalues. Hence H is indefinite so that the optimal value is $-\infty$.

c.

$$H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

Solution:
The characteristic polynomial is $p(\lambda) = \lambda[\lambda^2 - 8\lambda + 11]$ whose roots are $\lambda = 0,\ 4 \pm \sqrt{5}$. Hence H is positive semi-definite so that the set of all possible optimal solutions is the set of solutions to the equation $Hx + g = 0$ which is

$$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \quad \forall\, t \in \mathbb{R}.$$

(B) Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $g \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Does there exists a vector $u \in \mathbb{R}^3$ such that $f(tu) \overset{t\uparrow\infty}{\longrightarrow} -\infty$? If yes, construct u.

Solution:

The eigenvalues show that H is positive semi-definite with one zero eigenvalue. But the system $Hx + g$ is inconsistent, so no optimal solution exists. The vector $u = (-6, 1, 2)^T$ lies in the null-space of H, and $f(tu) = -4t$. Hence as $t \uparrow \infty$, $f(tu) \downarrow -\infty$.

(C) Consider the linearly constrained quadratic optimization problem

$$\mathcal{Q}(H, g, A, b) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Hx + g^T x \\ \text{subject to} & Ax = b \ , \end{array}$$

where H, A, g, and b are given by

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \ g = (1, 1, 1)^T, \ b = (4, 2)^T, \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

a. Compute a basis for the null space of A.
   Solution: A basis of $\mathrm{Nul}(A)$ is $(1, 0, -1)^T$.

b. Solve the problem $\mathcal{Q}(H, g, A, b)$ two different ways. One should use a basis for the null space of A and the other should not.

   Solution 1: Clearly, $\mathrm{rank}(A) = 3$ and H is positive definite by the determinant test. Therefore, by Part (a), the unique global solution can be found by solving $(\star)$:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 4 \\ 2 \end{pmatrix},$$

   which is given by

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 \\ 2 \\ 1 \\ -5 \\ -2 \end{pmatrix}.$$

   Solution 2: Again the solution is unique by Part (a). But now recall that the solution must be of the form

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

   since the vector $(1, 1, 1)^T$ solves $Ax = b$ and the vector $(1, 0, -1)$ spans the null space of A. Hence this is just a one dimensional problem in t which is solved by taking $t = \frac{1}{2}$.

(D) Let $H \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, $r \in \mathbb{R}^n \setminus \mathrm{Span}[\mathsf{e}]$, $\mu \in \mathbb{R}$. Solve the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Hx \\ \text{subject to} & \mathsf{e}^T x = 1 \quad \text{and} \quad r^T x = \mu \ , \end{array}$$

where $\mathsf{e} := (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ is the vector of all ones and $\mu \in \mathbb{R}$.

Solution:

$$\begin{aligned} Hx + ey + rz &= 0 \\ e^T x &= 1 \\ r^T z &= \mu \end{aligned}$$

Let

$$\begin{aligned} a &= e^T H^{-1} e \\ b &= e^T H^{-1} r \\ c &= r^T H^{-1} r, \end{aligned}$$

then

$$\begin{aligned} z &= \frac{-b + a\mu}{b^2 - ac} \\ y &= \frac{-1 - bz}{a} \\ x &= -H^{-1}(ey + rz) \end{aligned}$$