This exam will consist of three parts: (I) Linear Least Squares, (II) Quadratic Optimization, and (III) Optimality Conditions and Lagrangian Duality. The first two parts ((I) Linear Least Squares and (II) Quadratic Optimization) will have 2 multipart questions, and the third part ((III) Optimality Conditions and Lagrangian Duality) will have 3 multipart questions. This give a total of 7 questions is worth 50 points for a total of 350 points. The first two parts ((I) Linear Least Squares and (II) Quadratic Optimization) are identical to the two parts of the midterm exam, however, on the final, the questions will be taken from only two of the three question types ((i) theory, (ii) linear algebra, and (iii) computations). Please use the midterm exam study guide to prepare for these questions. A more detailed description of the third part of the final exam is given below.

III Optimality Conditions and Lagrangian Duality

1 Theory Question: For this question you will need to review all of the vocabulary words as well as the theorems from the notes on *Elements of Multivariable Calculus, Optimality Conditions for Unconstrained Problems*, and *Optimality Conditions for Constrained Optimization*. You may be asked to provide statements of first- and second-order optimality conditions for both constrained and unconstrained problems. In addition, you may be asked about the role of convexity in optimization, how it is detected, as well as first- and second-order conditions under which it is satisfied.

2 Computation: In this question you will be asked to compute gradients and Hessians, located and classify stationary points for specific optimizations problems, as well as test for the convexity of a problem. You may be asked to verify whether a function or set is convex.

3 Lagrangian Duality: In this problem you will be given a primal formulation of a convex optimization problem and then asked to compute its dual.

Sample Questions

(III) Optimality Conditions and Lagrangian Duality

Question 1: Theory Question

1. State the first- and second-order conditions for optimality for the following two problems:
   (a) Linear least squares: $\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax-b\|_2^2$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.
   (b) Quadratic Optimization: $\min_{x\in\mathbb{R}^n} \frac{1}{2}x^TQx + g^Tx$, where $Q \in \mathbb{R}^{n\times n}$ is symmetric and $g \in \mathbb{R}^n$.

2. Provide necessary and sufficient conditions under which a quadratic optimization problem be written as a linear least squares problem.

3. State the second-order necessary and sufficient optimality conditions for the problem $\min_{x\in\mathbb{R}^n} f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable.

4. State the first-order optimally conditions for the problem

$$\min_{x\in\Omega} f_0(x), \qquad (\spadesuit)$$

where

$$\Omega := \{x : f_i(x) \le 0, i = 1,\dots,s, f_i(x) = 0, i = s+1,\dots,m\}. \qquad (\clubsuit)$$

5. State the second-order sufficient conditions for optimality for the problem ($\spadesuit$) where Ω is given by ($\clubsuit$).

6. State first- and second-order necessary and sufficient conditions for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be convex.
7. Use a first-order necessary and sufficient condition for convexity to show that if $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function and $C \subset \mathbb{R}^n$ is a convex set, then $\bar{x}$ solves $\min_{x \in C} f(x)$ if and only if $\nabla f(\bar{x})^T (x - \bar{x}) \geq 0$ for all $x \in C$.

Question 2: Computation

1. If f_1 and f_2 are convex functions mapping $\mathbb{R}^n$ into $\mathbb{R}$, show that $f(x) := \max\{f_1(x), f_2(x)\}$ is also a convex function.
2. Let $F : \mathbb{R}^n \to \mathbb{R}^m$. Use the delta method to show that the gradient of the function $f(x) := \frac{1}{2}\|F(x)\|_2^2$ is
$$\nabla f(x) = \nabla F(x)^T F(x) \ .$$
3. A critical point of a function $f : \mathbb{R}^n \to \mathbb{R}$ is any point x at which $\nabla f(x) = 0$. Compute all of the critical points of the following functions. If no critical point exists, explain why.
   (a) $f(\bar{x}) = x_1^2 - 4x_1 + 2x_2^2 + 7$
   (b) $f(x) = e^{-\|x\|^2}$
   (c) $f(x) = x_1^2 - 2x_1 x_2 + \frac{1}{3}x_2^3 - 8x_2$
   (d) $f(x) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$
4. Show that the functions
$$f(x_1, x_2) = x_1^2 + x_2^3, \quad \text{and} \quad g(x_1, x_2) = x_1^2 + x_2^4$$
both have a critical point at $(x_1, x_2) = (0, 0)$ and that their associated Hessians are positive semi-definite. Then show that $(0, 0)$ is a local (global) minimizer for g but is not a local minimizer for f.
5. Find the local minimizers and maximizers for the following functions if they exist:
   (a) $f(x) = x^2 + \cos x$
   (b) $f(x_1, x_2) = x_1^2 - 4x_1 + 2x_2^2 + 7$
   (c) $f(x_1, x_2) = e^{-(x_1^2 + x_2^2)}$
   (d) $f(x_1, x_2, x_3) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$
6. Locate all of the KKT points for the following problems. Can you show that these points are local solutions? Global solutions?
   (a)
$$\begin{array}{ll} \text{minimize} & e^{(x_1 - x_2)} \\ \text{subject to} & e^{x_1} + e^{x_2} \leq 20 \\ & 0 \leq x_1 \end{array}$$
   (b)
$$\begin{array}{ll} \text{minimize} & e^{(-x_1 + x_2)} \\ \text{subject to} & e^{x_1} + e^{x_2} \leq 20 \\ & 0 \leq x_1 \end{array}$$
   (c)
$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 - 4x_1 - 4x_2 \\ \text{subject to} & x_1^2 \leq x_2 \\ & x_1 + x_2 \leq 2 \end{array}$$

(d)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x\|^2 \\ \text{subject to} & Ax = b \end{array}$$

where $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ satisfies $\mathrm{Nul}\,(A^T) = \{0\}$.

Question 3: Lagrangian Duality

1. Let $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$ and compute the Lagrangian dual to the problem

$$\begin{array}{lll} \mathcal{P} & \text{minimize} & c^T x \\ & \text{subject to} & Ax \le b,\ 0 \le x\ . \end{array}$$

2. Let $Q \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$ and compute the Lagrangian dual to the problem

$$\begin{array}{lll} \mathcal{P} & \text{minimize} & \frac{1}{2}x^T Q x + c^T x \\ & \text{subject to} & Ax \le b,\ 0 \le x\ . \end{array}$$

3. Let $Q \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Consider the optimization problem

$$\begin{array}{lll} \mathcal{P} & \text{minimize} & \frac{1}{2}x^T Q x + c^T x \\ & \text{subject to} & \|x\|_\infty \le 1\ . \end{array}$$

   (a) Show that this problem is equivalent to the problem

$$\hat{\mathcal{P}} \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Q x + c^T x \\ \text{subject to} & -e \le x \le e\ , \end{array}$$

   where e is the vector of all ones.

   (b) What is the Lagrangian for $\hat{\mathcal{P}}$?

   (c) Show that the Lagrangian dual for $\hat{\mathcal{P}}$ is the problem

$$\mathcal{D} \qquad \max -\frac{1}{2}(y-c)^T Q^{-1}(y-c) - \|y\|_1 \qquad = \qquad -\min \frac{1}{2}(y-c)^T Q^{-1}(y-c) + \|y\|_1\ .$$

   This is also the Lagrangian dual for $\mathcal{P}$.

4. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Consider the optimization problem

$$\begin{array}{lll} \mathcal{P} & \text{minimize} & \frac{1}{2}\|Ax - b\|_2^2 \\ & \text{subject to} & \|x\|_1 \le 1\ . \end{array}$$

   (a) Show that this problem is equivalent to the problem

$$\hat{\mathcal{P}} \qquad \begin{array}{ll} \text{minimize}_{(x,z,w)} & \frac{1}{2}\|w\|_2^2 \\ \text{subject to} & Ax - b = w, \\ & -z \le x \le z \ \text{ and }\ e^T z \le 1, \end{array}$$

   where e is the vector of all ones.

   (b) What is the Lagrangian for $\hat{\mathcal{P}}$?

   (c) Show that the Lagrangian dual for $\hat{\mathcal{P}}$ is the problem

$$\mathcal{D} \qquad \max -\frac{1}{2}\|y\|_2^2 - y^T b - \|A^T y\|_\infty \qquad = \qquad -\min \frac{1}{2}\|y-b\|_2^2 + \|A^T y\|_\infty - \frac{1}{2}\|b\|_2^2\ .$$

   This is also the Lagrangian dual for $\mathcal{P}$.