Math 408
Bond Basics: Fixed Income Securities

Bond is an example of a *financial instrument*. Other types of financial instruments of interest to us in this course are stocks, annuities, savings deposites, commercial paper, options, derivatives, mortgages, etc.... A bond is also an example of a *fixed-income security*. In general, a fixed-income security is a financial instrument having a predetermined and fixed cash flow stream and which is freely and easily traded. It should be emphasized though that this is not the case for all bonds, e.g. callable and adjustable bonds. In this section, we only discuss fixed-income financial instruments that have a predetermined and fixed cash flow stream.

Consider the cash flow stream for a bond having a fixed and uniform per period payment along with a one time final payment. The fixed and uniform per period payment is called the *coupon payment* for the bond (caution: sometimes the sum of one years worth of coupon payments is refered to as the coupon payment), the total time associated with the bond's cash flow stream is called the *time to maturity* of the bond, and the one time final payment is called the *face value* of the bond. A *zero coupon bond* is a bond whose coupon payment value is zero (i.e. there are no per period payments) and only has a face value. That is, the bond comes with one time only payment equal to its face value.

Every bond discussed in this section comes with a fixed time to maturity and period. The date on which the face value of the bond is payed is called the *date of maturity* of the bond. Most government bonds have a period of 6 months, but the time to maturity can vary from 3 months to 30 years. Bonds are typically sold at auction. Since Bonds can be traded (bought and sold) easily and freely, they are considered *securities*. In general, financial instruments that are easily and freely traded are called securities. Real estate is an example of a financial instrument that is not a security since it cannot be traded with the ease of a phone call. But there are financial instruments built on real estate that are securities such as Real Estate Investment Trusts, or REITs.

1. The Bond Price Formula and Yield

Since bonds are easily and freely traded, they must be priced. We now consider an elementary method for the pricing of bonds based on the notion of present value. Consider a bond having period $1/m$ years ($m = 12$ for months), coupon payments C/m (paid each period), time to maturity n periods, and face value F. The cash flow stream for this bond is then given by

$$x = (0, C/m, C/m, \ldots, C/m + F),$$

where this is a vector of length $n+1$ (coupon payments do not occur on the date of purchase).

Let us suppose that you can purchase this bond for the price P. What is the annual rate of return of this bond? It is reasonable to set the purchase price of the bond equal the present value of the bond. If we denote the annual rate of return of the bond by λ, then by our previous discussion on present value, we use the cash flow stream of the bond to get the formula

$$P = \frac{F}{(1+(\lambda/m))^n} + \sum_{k=1}^{n} \frac{C/m}{(1+(\lambda/m))^k}. \tag{1}$$

Thus, the annual rate of return of the bond is the value of λ that balances this equation (i.e. the value of λ that makes this equation valid). This annual rate of return is called the *yield*, or *yield to maturity*, of the bond. This formula can be simplified using properties of geometric series.

Taking $\rho = \frac{1}{(1+(\lambda/m))}$, we know that

$$\sum_{k=1}^{n} \rho^k = \rho \frac{1-\rho^n}{1-\rho} = \ .$$

By plugging this information into the Bond Price Formula (1) we have

$$\begin{aligned} P &= \frac{F}{(1+(\lambda/m))^n} + \sum_{k=1}^{n} \frac{C/m}{(1+(\lambda/m))^k} \\ &= \frac{F}{(1+(\lambda/m))^n} + \frac{C}{m} \frac{1}{(1+(\lambda/m))} \frac{[1-(1+(\lambda/m))^{-n}]}{1-(1+(\lambda/m))^{-1}} \\ &= F(1+(\lambda/m))^{-n} + \frac{C}{\lambda}[1-(1+(\lambda/m))^{-n}]. \end{aligned}$$

Definition 1.1. *(***Yield to Maturity and the Bond Price Formula***) The price of a bond having exactly n coupon periods remaining to maturity and a yield to maturity of λ, satisfies*

$$P = F(1+(\lambda/m))^{-n} + \frac{C}{\lambda}[1-(1+(\lambda/m))^{-n}]. \tag{2}$$

where P is the price of the bond, F is the face value, C is the yearly coupon payment, and m is the number of coupon payments per year.

Remark It is often the case that the annual coupon value is given as a percent of the face value, for example, $C = \tau F$. In this case, the formula (2) simplifies further to the formula

$$P = F\left[(1 - \frac{\tau}{\lambda})(1 + (\lambda/m))^{-n} + \frac{\tau}{\lambda}\right]. \tag{3}$$

We say that a bond is at *par* if $\tau = \lambda$ in which case $P = F$.

It is very important to note that the Bond Price Formula gives the price of the bond regardless of how far we have progressed into the cash flow stream of the bond from the date of issue.

Solving for λ in the Bond Price Formula is a bit tricky since it is nonlinear in λ. If we multiply the Bond Price Formula through by $\lambda(1 + (\lambda/m))^n$ we obtain the polynomial equation

$$0 = P\lambda(1 + (\lambda/m))^n - C[(1 + (\lambda/m))^n - 1] - F\lambda.$$

Numerous algorithms exist to compute the $n+1$ roots of this polynomial equation. The smallest positive root is the one taken for the yield to maturity (if a positive root exists!).

The yield to maturity of a bond is an valuable tool for comparing bonds. Riskier bonds should possess a yield to maturity that is greater than less risky bonds. It is the bond market that controls the yield. Tracking of the bond market is done through the yield to maturity. The yield to maturity is tied the market expectation for the rate of return on investment for differing levels of risk. For riskless investments such as treasure bills, the expected return on investment (or yield) is very low as compared with riskier bonds. This market expectation for the return on investment, or yield to maturity in the case of bonds, changes daily and closely tracks prevailing interests rates as well as the yields of other fixed income securities. It is the change in the yield to maturity that changes the purchase price of newly issued bonds, as well as for bonds that are well into their lifetime cash flow stream. Indeed, an "old" bond behaves in exactly the same way as a "new" bond from the perspective of the purchaser. The bonds differ in the case where the purchase date lies between coupon dates. In this case an adjusted interim coupon payment must be paid to the seller of the bond equal to the percentage of the coupon payment corresponding to the percentage of the coupon payment period that has elapsed since the last coupon payment date and the date of purchase.

As the yield to maturity changes so does the purchase price of new and old bond issues. Therefore, it is important to understand how bond prices change as the yield changes. For example, if you need to sell a bond before its maturity date, you would prefer to sell it at a present value that is greater than the present value of the bond when computed using the yield on the date of purchase since by doing so your effective rate of return on the bond is greater than on the date of purchase. Note that as the yield to maturity (or rate of return) of a bond increases, its purchase price (or present value) decreases. Correspondingly, as the yield to maturity of a bond decreases, its purchase price increases. For this reason, the bond market is said to be "up" when the yield to maturity goes down, and the bond market is said to "down" when the yield to maturity goes up. Since prevailing interest rates are tied to yields, the bond market typically goes up as interest rates fall. One need only look at a graph of bond purchase prices between the years 2000 and 2003 to see this effect.

Example 1.2. *The annual coupon payment of a bond is typically given as a percentage of its face value. A* 5 *percent bond is one whose total annual coupon payments equal* 5 *percent of the face value of the bond. The purchase price of a* 5 *percent semi-annual 30 year bond having a* 4 *percent yield to maturity is*

$$\begin{aligned} P &= \frac{F}{(1.02)^{60}} + \frac{.05F}{.04}[1 - (1.02)^{-60}] \\ &= F\left[1.25 - 0.25(1.02)^{-60}\right] \\ &= 1.1738F. \end{aligned}$$

That is, if the face value of the bond is $1,000, *then you need to pay* $1,173.80 *for each such bond.*

Now suppose that two days after this bond is purchased the yield jumps to 5.5 *percent. The purchase price of this bond is now*

$$\begin{aligned} P &= \frac{F}{(1.0275)^{60}} + \frac{.05F}{.055}[1 - (1.0275)^{-60}] \\ &= F\left[0.9091 + 0.0909(1.0275)^{-60}\right] \\ &= 0.9270F. \end{aligned}$$

That is, if the face value of the bond is $1,000, *then you need to pay* $927 *for each such bond. On the other hand, if you for some reason you need to sell the bond you just purchased two days earlier, then you have just lost* $246.80 *on each of these bonds!*

What would be the per bond loss if this were a 5 *year bond with the same characteristics rather than a* 30 *year bond?*

Thus, each bond comes with a certain amount of risk even though it is a fixed income security. This risk is tied to fluxuations in the yield, or correspondingly, fluxuations in the prevailing interest rates.

It is important to understand this risk. One tool for understanding this risk is to measure the sensitivity of the bond price to its yield, or equivalently, what is the instantaneous rate of change of the bond price with respect to yield. Again, this question is answered in the calculus. It is given by the derivative of the bond price with respect to yield. Differentiating the Bond Price Formula (3) with respect to λ gives

$$\frac{dP}{d\lambda} = F\lambda^{-1}(1+\lambda/m)^{-n}[\frac{n}{m}(\tau-\lambda)(1+\lambda/m)^{-1} - \frac{\tau}{\lambda}((1+\lambda/m)^{n}-1)],$$

where the annual coupon value is τF.

2. Duration

In the finance liturature the standard measure of the sensitivity of price to yield is known as the *duration* of a bond. The notion of duration can be attached to any cash flow stream once a prevailing rate of return is agreed upon. Let $(x_0, x_1, \ldots, x_n)$ be a cash flow stream where the flow element x_k occurs at time t_k. If we denote by $PV(t_k)$ the present value of x_k, then the durations of the cash flow stream is

$$D = \frac{PV(t_0)t_0 + PV(t_1)t_1 + \cdots + PV(t_n)t_n}{PV},$$

where

$$PV = PV(t_0) + PV(t_1) + \cdots + PV(t_n)$$

is the present value of the entire cash flow stream. Note that if there is only one time period t_1, then the duration must equal the length t_1. This highlights the fact that the units of D are the same as those of the times t_k.

In the case of a bond with m evenly spaced coupon payments per year, yield λ, and a payment of c_k in the kth period, we obtain the *Macaulay* duration of the bond:

$$D = \frac{\sum_{k=1}^{n}(k/m)c_k/[1+(\lambda/m)]^k}{PV},$$

where

$$PV = \sum_{k=1}^{n} c_k/[1+(\lambda/m)]^k \ .$$

If the coupon payment is cF where F is the face value of the bond, then this expression for duration simplifies to

$$\begin{aligned} D &= \frac{\frac{nF}{m[1+(\lambda/m)]^n} + \frac{cF}{m}\sum_{k=1}^n \frac{k}{[1+(\lambda/m)]^k}}{F[1+(\lambda/m)]^{-n} + cF\sum_{k=1}^n [1+(\lambda/m)]^{-k}} \\ &= \frac{\frac{n}{m[1+(\lambda/m)]^n} + \frac{c}{m}\sum_{k=1}^n \frac{k}{[1+(\lambda/m)]^k}}{[1+(\lambda/m)]^{-n} + c\sum_{k=1}^n [1+(\lambda/m)]^{-k}} \\ &= \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my}, \end{aligned}$$

where $y = \lambda/m$. A remarkable property of duration is that it is related to the derivative of the price with respect to yield.

Theorem 2.1 (Price Sensitivity Formula)**.** *The derivative of price P with respect to yield of a fixed income security is*

$$\frac{dP}{d\lambda} = -D_M P,$$

where

$$D_M = \frac{D}{1+(\lambda/m)}$$

is the modified duration of the security.

Plugging this information into our previous formulas we obtain the following.

Corollary 2.2. *(***Bond Duration** *) The duration D of a bond having exactly n coupon periods remaining to maturity and a yield to maturity of λ is given by*

$$D = -\frac{(1+\lambda/m)}{P}\frac{dP}{d\lambda} = \frac{(1+\lambda/m)}{\lambda} - \frac{(1+\lambda/m) + \frac{n}{m}(\tau - \lambda)}{\lambda + \tau(((1+\lambda/m)^n - 1)}, \tag{4}$$

where P is the price of the bond, F is the face value, τF is the yearly coupon payment as a percentage of face value, and m is the number of coupon payments per year.

3. The Duration of a Portfolio

A bond portfolio is simply a basket of bonds grouped together. The cash flow of a bond portfolio at a particular moment in time is the sum of the cash flows of the portfolios. In this way we can construct a cash flow stream for the portfolio as a whole and compute its corresponding present value. Due to the additive nature if the individual cash flows in a portfolio, the present value of the portfolio is simply the sum of the present values of the individual bonds. It is also easy to compute the duration of a portfolio from the durations of each individual bond in the portfolio. Indeed, if we let PV_k^j denote the present value of the cash flow from bond j in period k at time t_k, then the duration of bond j is

$$D^j = \frac{\sum_{k=1}^n t_k PV_k^j}{PV^j},$$

where

$$PV^j = \sum_{k=1}^n PV_k^j$$

is the present value of bond j. If there are N bonds in the portfolio, then

$$\begin{aligned} PV^1D^1 + PV^2D^2 + \cdots + PV^ND^N &= \sum_{j=1}^N \left[\sum_{k=1}^n t_k PV_k^j\right] \\ &= \sum_{k=1}^n t_k \left[\sum_{j=1}^N PV_k^j\right] \\ &= \sum_{k=1}^n t_k PV^j \ . \end{aligned}$$

Therefore, the duration of the portfolio is given by

$$\begin{aligned} D &= \frac{\sum_{k=1}^n t_k PV^j}{PV} \\ &= \frac{\sum_{j=1}^N PV^j D^j}{PV} \\ &= \sum_{j=1}^N w_j D^j, \end{aligned}$$

where the weights w_j are given by

$$w_j = \frac{PV^j}{PV} \qquad \text{for } j = 1, 2, \ldots, N.$$

4. Immunization

Immunization is a term used in finance that refers a technique that *immunizes* a portfolio of fixed income securities against the investment risk associated with fluxuations in yield, or interest rate risk.

A common objective in the construction of a portfolios of fixed income securities is to construct the cash flow stream so that certain financial obligations are met in the future. If we try to do this with fixed income securities, then we face the possibility of not being able to meet our obligations due to unfavorable yield fluxuations. A method for mitigating this risk is to not only match the cash flow streams to our obligations, but to also match the duration associated with this stream. We illustrate this with a simple example.

A corporation has a debt of 1 million dollars that it is obliged to pay in 10 years time (the duration of this debt is 10 years, why?). It is decided that the debt is to be payed by purchasing a portfolio of bonds. The bonds are purchased today and then sold on the day that the debt must be paid. The bonds to be included in the portfolio all have a face value of $100 and a 1 year period. They are described in the following table.

Bond Choices

	Rate	**Maturity**	**Price**	**Yield**	**Duration**
Bond 1	6%	30 yr	$69.04	9.00%	11.88
Bond 2	11%	10 yr	$113.01	9.00%	6.33
Bond 3	9%	20 yr	$100.00	9.00%	9.95

The durations for these bonds are computed using the Macaulay duration formula. For example, to compute the duration for bond 1 we have

$$D_1 = \frac{1.09}{0.09} - \frac{1.09 + 30[.06 - .09]}{.06[(1.09)^30 - 1] + .09} = 11.88 \ .$$

The portfolio is to be chosen to match both the present value of the debt and its duration. That is, if V_i is the amount of type i bond to purchase, then we must solve the equations

$$\begin{aligned} PV &= V_1 + V_2 + V_3 \\ 10PV &= D_1 V_1 + D_2 V_2 + D_3 V_3 \end{aligned}$$

One solution is to just use bonds 1 and 2 which gives

$$V_1 = \$299143.72 \quad \text{and} \quad V_2 = \$123267.08 \ .$$