The Term Structure of Interest Rates
Spot Rates and Discount Factors

The prevailing market for money effects the annual interest rates at which money is either borrowed or lent. All things being equal, we have seen from our quick study of bonds, that money lent for longer terms (i.e. longer periods of time) has associated with it greater risk. Hence the annual interest rate typically increases with the term that the money is to be borrowed. Many factors go into the determination of the annual interest rate associated with a given transaction, but essentially all deviations from prevailing, or average, rates are associated with those aspects the transaction that either increase or decrease the markets perception of the *risk* involved. The rates go up as the perceived risk increases. In our present discussion, we only consider the risk associated with term structure. We describe the annual interest rate for a loan having term t as a function of the time t:

$$s(t) \quad = \quad \text{annual interest rate for a loan with term } t.$$

Here, time $t = 0$ denotes the present time. One can think of time t as being measured continously, or as being a discrete measure of time where, for example, $t =$ days, months, or years. One often takes the discrete measure of time to coincide with the compounding structure associated with the annual rate under consideration.

We say that the function $s(t)$ gives the *spot rates* that define the *term structure* of money being lent or borrowed. Again, the spot rate $s(t)$ is the rate of interest, expressed in yearly terms, charged for money held from the present ($t = 0$) until time t. Typically, $s(t)$ is an increasing function of t and is concave down. The definition of the spot rate implicitly assumes a compounding convention. Examples of typical compounding conventions are as follows:

Yearly: Under yearly compounding, the spot rate is defined so that

$$(1 + s(t))^t$$

is the factor by which a deposit held for t years will grow (here, t must be an integer, or an adjustment must be made).

m **periods per year:** Under compounding m periods per year, the spot rate is defined so that

$$(1 + s(t)/m)^{mt}$$

is the factor by which a deposit held for t years will grow (here, mt must be an integer, or an adjustment must be made).

Continuous: Under continuous compounding, the spot rate is defined so that

$$e^{s(t)t}$$

is the factor by which a deposit held for t years will grow (here, t can be any real number).

Having the spot rates, we can define the associated *discount factors* $d(t)$ as another function of time. These are the factors by which future cash flows are multiplied to obtain the equivalent present value. For the various compounding conventions they are given as follows:

Yearly: Under yearly compounding,

$$d(t) = (1 + s(t))^{-t}$$

is the factor by which a cash flow received in t years must be multiplied to determine its present value (here, t must be an integer, or an adjustment must be made).

m **periods per year:** Under compounding m periods per year,

$$d(t) = (1 + s(t)/m)^{-mt}$$

is the factor by which a cash flow received in t years must be multiplied to determine its present value (here, mt must be an integer, or an adjustment must be made).

Continuous: Under continuous compounding,

$$d(t) = e^{-s(t)t}$$

is the factor by which a cash flow received in t years must be multiplied to determine its present value (here, t can be any real number).

Discount factors transform future cash flows directly into an equivalent present value. Hence, given a cash flow stream

$$x = (x_0, x_1, \ldots, x_n),$$

the present value of this stream, relative to prevailing spot rates, is

$$PV = x_0 + d(1)x_1 + d(2)x_2 + \ldots + d(n)x_n.$$

Thus, the discount factor $d(k)$ acts like a *price* for the cash received at time k. The *value* of the stream is then determined by adding up *price times quantity* for all of the cash flow components of the stream.