Math 408A
Convexity

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Definition: *A set $C \subset \mathbb{R}^n$ is said to be convex if for every $x, y \in C$ and $\lambda \in [0, 1]$ one has*

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$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2).$$

The function f *is said to be strictly convex if for every two distinct points* $x_1, x_2 \in \mathbb{R}^n$ *and* $\lambda \in [0,1]$ *we have*

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2).$$

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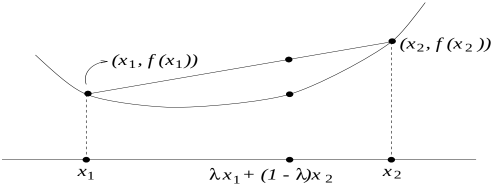

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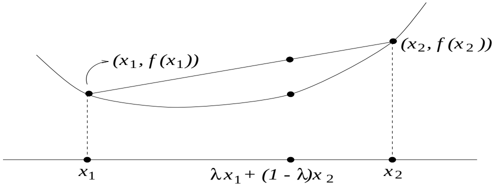

That is, the set

$$\mathrm{epi}\,(f) = \{(x, \mu) \,:\, f(x) \le \mu\},$$

called the *epi-graph* of f, is a convex set.

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Examples of convex functions are

$$c^T x, \quad \|x\|, \quad x^T H x \quad \text{when } H \text{ is psd.}$$

Convexity and Optimality

Theorem: *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex. If $\overline{x} \in \mathbb{R}^n$ is a local minimum for f, then $\overline{x}$ is a global minimum for f. If, in addition, f is strictly convex, then $\bar{x}$ is the unique global minimizer.*

Convexity and Optimality

Proof: Suppose $f(\widehat{x}) < f(\overline{x})$. Let $\epsilon > 0$ be such that

$$f(\overline{x}) \leq f(x) \quad \text{whenever} \quad \|x - \overline{x}\| \leq \epsilon \quad \text{and} \quad \epsilon < 2\|\overline{x} - \widehat{x}\| \ .$$

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Set $\lambda := \epsilon(2\|\overline{x} - \widehat{x}\|)^{-1} < 1$ and $x_\lambda := \overline{x} + \lambda(\widehat{x} - \overline{x})$.

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Then $\|x_\lambda - \overline{x}\| \leq \epsilon/2$ and

$$f(x_\lambda) \leq (1 - \lambda)f(\overline{x}) + \lambda f(\widehat{x}) < f(\overline{x}).$$

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$$f(x_\lambda) \le (1-\lambda)f(\overline{x}) + \lambda f(\widehat{x}) < f(\overline{x}).$$

This contradicts the choice of ϵ and so no such $\widehat{x}$ exists.

Directional Derivatives and Convexity

Lemma: *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex (not necessarilty differentiable).*

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1. *Given $x, d \in \mathbb{R}^n$ the difference quotient*

$$\frac{f(x + td) - f(x)}{t}$$

is a non-decreasing function of t on $(0, +\infty)$.

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2. *For every $x, d \in \mathbb{R}^n$ the directional derivative $f'(x; d)$ always exists and is given by*

$$f'(x; d) := \inf_{t>0} \frac{f(x + td) - f(x)}{t}.$$

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Now if the difference quotient on the right is non-decreasing in t on $(0, +\infty)$, then the limit is necessarily given by the infimum of these ratios.
This infimum always exists and so $f'(x; d)$ always exists and is given by the infimum.

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$$\begin{aligned} f(x+t_1 d) &= f\left(x + \left(\frac{t_1}{t_2}\right) t_2 d\right) \\ &= f\left[\left(1-\left(\frac{t_1}{t_2}\right)\right)x + \left(\frac{t_1}{t_2}\right)(x+t_2 d)\right] \end{aligned}$$

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Hence

$$\frac{f(x+t_1 d) - f(x)}{t_1} \le \frac{f(x+t_2 d) - f(x)}{t_2}.$$

The Subdifferential Inequality

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then for all $x \in \mathrm{dom}\,(f)$ and $y \in \mathbb{R}^n$,

$$f'(x; (y - x)) = \inf_{t>0} \frac{f(x + t(y - x)) - f(x)}{t} \; .$$

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We will call this inequality the *subdifferential inequality*.

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$$f(y) \geq f(x) + \nabla f(x)^T (y-x) \qquad \forall\, y \in \mathbb{R}^n \; .$$

This inequality states that the tangent plane to the graph of f lies entirely below the graph of f.

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Theorem: *Let* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ *be convex (not necessarilty differentiable) and let* $\bar{x} \in \mathrm{dom}\,(f)$*. Then the following three statements are equivalent.*

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Theorem: *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex (not necessarilty differentiable) and let $\bar{x} \in \mathrm{dom}\,(f)$. Then the following three statements are equivalent.*

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Proof: (i)⇒(ii): Done.
(ii)⇒(iii): Follows from the subdifferential inequality;

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(iii)⇒(i): Obvious.

Convexity and Optimality Conditions

Theorem: *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and suppose that $\bar{x} \in \mathbb{R}^n$ is a point at which f is differentiable. Then $\bar{x}$ is a global minimum of f if and only if $\nabla f(\bar{x}) = 0$.*

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If f *is differentiable on* $\mathbb{R}^n$, *then the following statements are equivalent.*

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(3) $(\nabla f(x) - \nabla f(y))^T(x - y) \geq 0$ *for all* $x, y \in \mathbb{R}^n$.

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(3) *$(\nabla f(x) - \nabla f(y))^T(x - y) \geq 0$ for all $x, y \in \mathbb{R}^n$.*

If f is twice differentiable then f is convex if and only if $\nabla^2 f(x)$ is positive semi-definite for all $x \in \mathbb{R}^n$.

$f(y) \geq f(x) + \nabla f(x)^T(y-x) \ \forall x, y \in \mathbb{R}^n \ \Rightarrow$ $(\nabla f(x) - \nabla f(y))^T(x-y) \geq 0 \ \forall \ x, y \in \mathbb{R}^n$

For all $x, y \in \mathbb{R}^n$, the subdifferential inequality gives

$$\begin{aligned} f(y) &\geq f(x) + \nabla f(x)^T(y-x) \\ f(x) &\geq f(y) + \nabla f(y)^T(x-y) \end{aligned}$$

Adding these two inequalities and canceling like terms from each side gives

$$0 \geq \nabla f(x)^T(y-x) + \nabla f(y)^T(x-y)$$

or equivalently

$$0 \leq (\nabla f(x) - \nabla f(y))^T(x-y) \ .$$

$(\nabla f(x) - \nabla f(y))^T(x-y) \geq 0 \;\forall\; x, y \in \mathbb{R}^n \;\Rightarrow$ $f(y) \geq f(x) + \nabla f(x)^T(y-x) \;\forall x, y \in \mathbb{R}^n$

Let $x, y \in \mathbb{R}^n$. By the MVT there exists $0 < \lambda < 1$ such that

$$f(y) - f(x) = \nabla f(x_\lambda)^T(y-x)$$

where $x_\lambda := \lambda y + (1-\lambda)x$. Then

$$\begin{aligned} 0 &\leq [\nabla f(x_\lambda) - \nabla f(x)]^T(x_\lambda - x) \\ &= \lambda[\nabla f(x_\lambda) - \nabla f(x)]^T(y-x) \\ &= \lambda[f(y) - f(x) - \nabla f(x)^T(y-x)]. \end{aligned}$$

Hence $f(y) \geq f(x) + \nabla f(x)^T(y-x)$.

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Let $x, y \in \mathbb{R}^n$ and set

$$\alpha := \max_{\lambda \in [0,1]} \varphi(\lambda) := [f(\lambda y + (1-\lambda)x) - (\lambda f(y) + (1-\lambda) f(x))].$$

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Show $\alpha \leq 0$.

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$[0,1]$ is compact and φ continuous, so $\exists\ \lambda \in [0,1]$ such that $\varphi(\lambda) = \alpha$.

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$$0 = \varphi'(\lambda) = \nabla f(x_\lambda)^T(y-x) + f(x) - f(y)$$

where $x_\lambda = x + \lambda(y-x)$, or equivalently

$$\lambda(f(y) - f(x)) = -\nabla f(x_\lambda)^T(x - x_\lambda).$$

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where $x_\lambda = x + \lambda(y-x)$, or equivalently

$$\lambda(f(y) - f(x)) = -\nabla f(x_\lambda)^T(x - x_\lambda).$$

Hence, by the subdifferential inequality,

$$\begin{aligned} \alpha &= f(x_\lambda) - (f(x) + \lambda(f(y) - f(x))) \\ &= f(x_\lambda) + \nabla f(x_\lambda)^T(x - x_\lambda) - f(x) \\ &\leq 0 \end{aligned}$$

f convex $\Rightarrow \nabla^2 f(x)$ PSD on $\mathbb{R}^n$

If f is convex,

$$f(x+td) \geq f(x) + t\nabla f(x)^T d \qquad \forall\ t \in \mathbb{R},\ x, d \in \mathbb{R}^n.$$

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So

$$f(x) + t\nabla f(x)^T d + \frac{t^2}{2} d^T \nabla^2 f(x) d + o(t^2) \geq f(x) + t\nabla f(x)^T d$$

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If f is convex,

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For $x, y \in \mathbb{R}^n$, the MVT implies there exists $\lambda \in (0,1)$ such that

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x_\lambda)(y - x)$$

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Therefore, f is convex.

Checking for Positive Definiteness

Theorem: *Let $H \in \mathbb{R}^{n \times n}$ be symmetric. We define the kth principal minor of H, denoted $\Delta_k(H)$, to be the determinant of the upper left-hand $k \times k$ submatrix of H. Then H is positive definite if and only if $\Delta_k(H) > 0$ for $k = 1, 2, \ldots, n$.*

Checking for Positive Definiteness

Example:

$$H = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 5 & 1 \\ -1 & 1 & 4 \end{bmatrix}.$$

We have

$$\Delta_1(H) = 1, \quad \Delta_2(H) = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4, \quad \Delta_3(H) = \det(H) = 8.$$

Therefore, H is positive definite.

Sublinear Functionals

Definition: *Let* $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. *We say that h is positively homogeneous if*

$$h(\lambda x) = \lambda h(x) \quad \text{for all } x \in \mathbb{R} \text{ and } \lambda > 0.$$

We say that h is subadditive if

$$h(x + y) \leq h(x) + h(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Finally, we say that h is sublinear if it is both subadditive and positively homogeneous.

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Note: Sublinear functionals can be thought of as a generalization of the notion of a norm. In particular, every norm is a sublinear functional.

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Lemma: *A function is sublinear if and only if its epigraph is a convex cone.*

Directional Derivatives of Convex Functions are Sublinear

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Thus, in particular, the function $f'(x; \cdot)$ *is a convex function.*

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Proof: (Positive Homogeneity) Let $x \in \operatorname{dom}(f)$, $d \in \mathbb{R}^n$, and $\lambda > 0$. Then

$$f'(x; \lambda d) = \lim_{t \downarrow 0} \frac{f(x + t\lambda d) - f(x)}{t}$$

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Examples of Convex Functions

Set $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n \; | 0 < x_i, \;\; i = 1, \ldots, n\}$ and $\mathbb{S}^n_{++}$ equal to the set on $n \times n$ symmetric positive definite matrices.

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