Math 408A: Linear Algebra Review

Matrices in $\mathbb{R}^{m\times n}$

$A \in \mathbb{R}^{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrices in $\mathbb{R}^{m\times n}$

$A \in \mathbb{R}^{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet 1} & a_{\bullet 2} & \dots & a_{\bullet n} \end{bmatrix}$$

columns

Matrices in $\mathbb{R}^{m\times n}$

$A \in \mathbb{R}^{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet 1} & a_{\bullet 2} & \dots & a_{\bullet n} \end{bmatrix} = \begin{bmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{bmatrix}$$

columns rows

Matrices in $\mathbb{R}^{m\times n}$

$A \in \mathbb{R}^{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet 1} & a_{\bullet 2} & \dots & a_{\bullet n} \end{bmatrix} = \begin{bmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{bmatrix}$$

columns rows

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

Matrices in $\mathbb{R}^{m\times n}$

$A \in \mathbb{R}^{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet 1} & a_{\bullet 2} & \dots & a_{\bullet n} \end{bmatrix} = \begin{bmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{bmatrix}$$

columns rows

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet 1}^T \\ a_{\bullet 2}^T \\ \vdots \\ a_{\bullet n}^T \end{bmatrix}$$

Matrices in $\mathbb{R}^{m\times n}$

$A \in \mathbb{R}^{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet 1} & a_{\bullet 2} & \dots & a_{\bullet n} \end{bmatrix} = \begin{bmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{bmatrix}$$

columns rows

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet 1}^T \\ a_{\bullet 2}^T \\ \vdots \\ a_{\bullet n}^T \end{bmatrix} = \begin{bmatrix} a_{1\bullet}^T & a_{2\bullet}^T & \dots & a_{m\bullet}^T \end{bmatrix}$$

Matrix Vector Multiplication

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix Vector Multiplication

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Matrix Vector Multiplication

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$$

Matrix Vector Multiplication

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Matrix Vector Multiplication

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= x_1\, a_{\bullet 1} \;+\; x_2\, a_{\bullet 2} \;+\; \cdots \;+\; x_n\, a_{\bullet n}$$

Matrix Vector Multiplication

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= x_1 \, a_{\bullet 1} \; + \; x_2 \, a_{\bullet 2} \; + \; \cdots \; + \; x_n \, a_{\bullet n}$$

A linear combination of the columns.

The Range of a Matrix

Let $A \in \mathbb{R}^{m \times n}$ (an $m \times n$ matrix having real entries).

The Range of a Matrix

Let $A \in \mathbb{R}^{m \times n}$ (an $m \times n$ matrix having real entries).

Range of A

$$\mathrm{Ran}(A) \;=\; \{y \in \mathbb{R}^m \mid \exists\, x \in \mathbb{R}^n \text{ such that } y = Ax\}$$

The Range of a Matrix

Let $A \in \mathbb{R}^{m \times n}$ (an $m \times n$ matrix having real entries).

Range of A

$$\mathrm{Ran}(A) \; = \; \{y \in \mathbb{R}^m \; | \exists \, x \in \mathbb{R}^n \text{ such that } y = Ax\}$$

$$\mathrm{Ran}(A) = \text{ the linear span of the columns of } A$$

A Couple of Special Subspaces

Let $v_1, \dots, v_k \in \mathbb{R}^n$.

A Couple of Special Subspaces

Let $v_1, \ldots, v_k \in \mathbb{R}^n$.

- The linear span of $v_1, \ldots, v_k$:

$$\mathrm{Span}(v_1, \ldots, v_k) = \{y \mid y = \xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_k v_k,\ \xi_1, \ldots, \xi_k \in \mathbb{R}\}$$

A Couple of Special Subspaces

Let $v_1, \ldots, v_k \in \mathbb{R}^n$.

- The linear span of $v_1, \ldots, v_k$:

$$\mathrm{Span}(v_1, \ldots, v_k) = \{y \mid y = \xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_k v_k,\ \xi_1, \ldots, \xi_k \in \mathbb{R}\}$$

- The subspace orthogonal to $v_1, \ldots, v_k$:

$$\{v_1, \ldots, v_k\}^{\perp} = \{z \in \mathbb{R}^n \mid z \bullet v_i = 0,\ \ i = 1, \ldots, k\}$$

A Couple of Special Subspaces

Let $v_1, \ldots, v_k \in \mathbb{R}^n$.

- The linear span of $v_1, \ldots, v_k$:

$$\mathrm{Span}(v_1, \ldots, v_k) = \{y \mid y = \xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_k v_k,\ \xi_1, \ldots, \xi_k \in \mathbb{R}\}$$

- The subspace orthogonal to $v_1, \ldots, v_k$:

$$\{v_1, \ldots, v_k\}^\perp = \{z \in \mathbb{R}^n \mid z \bullet v_i = 0,\ \ i = 1, \ldots, k\}$$

Facts:

A Couple of Special Subspaces

Let $v_1, \ldots, v_k \in \mathbb{R}^n$.

- The linear span of $v_1, \ldots, v_k$:

$$\mathrm{Span}(v_1, \ldots, v_k) = \{y \;|y = \xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_k v_k,\; \xi_1, \ldots, \xi_k \in \mathbb{R}\}$$

- The subspace orthogonal to $v_1, \ldots, v_k$:

$$\{v_1, \ldots, v_k\}^{\perp} = \{z \in \mathbb{R}^n \;|z \bullet v_i = 0,\; i = 1, \ldots, k\}$$

Facts:

$$\{v_1, \ldots, v_k\}^{\perp} = \mathrm{Span}(v_1, \ldots, v_k)^{\perp}$$

A Couple of Special Subspaces

Let $v_1, \ldots, v_k \in \mathbb{R}^n$.

- The linear span of $v_1, \ldots, v_k$:

$$\mathrm{Span}(v_1, \ldots, v_k) = \{y \mid y = \xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_k v_k,\ \xi_1, \ldots, \xi_k \in \mathbb{R}\}$$

- The subspace orthogonal to $v_1, \ldots, v_k$:

$$\{v_1, \ldots, v_k\}^{\perp} = \{z \in \mathbb{R}^n \mid z \bullet v_i = 0,\ \ i = 1, \ldots, k\}$$

Facts:

$$\{v_1, \ldots, v_k\}^{\perp} = \mathrm{Span}(v_1, \ldots, v_k)^{\perp}$$

$$\left[\mathrm{Span}(v_1, \ldots, v_k)^{\perp}\right]^{\perp} = \mathrm{Span}(v_1, \ldots, v_k)$$

Matrix Vector Multiplication

A row space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1\bullet} \bullet x \\ a_{2\bullet} \bullet x \\ \vdots \\ a_{m\bullet} \bullet x \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i} x_i \\ \sum_{i=1}^n a_{2i} x_i \\ \vdots \\ \sum_{i=1}^n a_{mi} x_i \end{bmatrix}$$

The dot product of x with the rows of A.

The Null Space of a Matrix

Let $A \in \mathbb{R}^{m \times n}$ (an $m \times n$ matrix having real entries).

The Null Space of a Matrix

Let $A \in \mathbb{R}^{m \times n}$ (an $m \times n$ matrix having real entries).

Null Space of A

$$\mathrm{Nul}(A) \; = \; \{x \in \mathbb{R}^n \, | Ax = 0\}$$

The Null Space of a Matrix

Let $A \in \mathbb{R}^{m\times n}$ (an $m \times n$ matrix having real entries).
Null Space of A

$$\mathrm{Nul}(A) \;=\; \{x \in \mathbb{R}^n \;|Ax = 0\,\}$$

$$\mathrm{Nul}(A) \quad = \quad \text{subspace orthogonal to the rows of } A$$

The Null Space of a Matrix

Let $A \in \mathbb{R}^{m\times n}$ (an $m \times n$ matrix having real entries).
Null Space of A

$$\mathrm{Nul}(A) \ = \ \{x \in \mathbb{R}^n \ | Ax = 0\}$$

$$\begin{aligned} \mathrm{Nul}(A) &= \text{subspace orthogonal to the rows of } A \\ &= \mathrm{Span}(a_{1\bullet}, a_{2\bullet}, \ldots, a_{m\bullet})^{\perp} \end{aligned}$$

The Null Space of a Matrix

Let $A \in \mathbb{R}^{m\times n}$ (an $m \times n$ matrix having real entries).

Null Space of A

$$\mathrm{Nul}(A) = \{x \in \mathbb{R}^n \,|Ax = 0\}$$

$$\begin{aligned}\mathrm{Nul}(A) &= \text{subspace orthogonal to the rows of } A \\ &= \mathrm{Span}(a_{1\bullet}, a_{2\bullet}, \ldots, a_{m\bullet})^{\perp} \\ &= \mathrm{Ran}(A^T)^{\perp}\end{aligned}$$

The Null Space of a Matrix

Let $A \in \mathbb{R}^{m\times n}$ (an $m \times n$ matrix having real entries).
Null Space of A

$$\mathrm{Nul}(A) \;=\; \{x \in \mathbb{R}^n \;|Ax = 0\}$$

$$\begin{aligned} \mathrm{Nul}(A) &= \text{subspace orthogonal to the rows of } A \\ &= \mathrm{Span}(a_{1\bullet}, a_{2\bullet}, \ldots, a_{m\bullet})^{\perp} \\ &= \mathrm{Ran}(A^T)^{\perp} \end{aligned}$$

Fundamental Theorem of the Alternative:

$$\mathrm{Nul}(A) = \mathrm{Ran}(A^T)^{\perp} \qquad \mathrm{Ran}(A) = \mathrm{Nul}(A^T)^{\perp}$$

Block Structured Matrices

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

Block Structured Matrices

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right]$$

Block Structured Matrices

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right] = \left[\begin{array}{c|c} B & I_{3\times 3} \\ \hline 0_{2\times 3} & C \end{array}\right]$$

Block Structured Matrices

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right] = \left[\begin{array}{c|c} B & I_{3\times 3} \\ \hline 0_{2\times 3} & C \end{array}\right]$$

where

$$B = \begin{bmatrix} 3 & -4 & 1 \\ 2 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

Multiplication of Block Structured Matrices

Consider the matrix product AM, where

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}$$

Multiplication of Block Structured Matrices

Consider the matrix product AM, where

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}$$

Can we exploit the structure of A?

Multiplication of Block Structured Matrices

Consider the matrix product AM, where

$$A = \left[\begin{array}{rrr|rrr} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right] \quad \text{and} \quad M = \left[\begin{array}{rr} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ \hline 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{array}\right]$$

Can we exploit the structure of A?

Multiplication of Block Structured Matrices

Consider the matrix product AM, where

$$A = \left[\begin{array}{rrr|rrr} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right] \quad \text{and} \quad M = \left[\begin{array}{rr} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ \hline 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{array}\right]$$

Can we exploit the structure of A?

$$A = \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix}$$

Multiplication of Block Structured Matrices

Consider the matrix product AM, where

$$A = \left[\begin{array}{rrr|rrr} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right] \quad \text{and} \quad M = \left[\begin{array}{rr} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ \hline 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{array}\right]$$

Can we exploit the structure of A?

$$A = \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \quad \text{so take} \quad M = \begin{bmatrix} X \\ Y \end{bmatrix},$$

Multiplication of Block Structured Matrices

Consider the matrix product AM, where

$$A = \left[\begin{array}{rrr|rrr} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right] \quad \text{and} \quad M = \left[\begin{array}{rr} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ \hline 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{array}\right]$$

Can we exploit the structure of A?

$$A = \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \quad \text{so take} \quad M = \begin{bmatrix} X \\ Y \end{bmatrix},$$

$$\text{where} \quad X = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}.$$

Multiplication of Block Structured Matrices

$$AM = \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Multiplication of Block Structured Matrices

$$AM = \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix}$$

Multiplication of Block Structured Matrices

$$
\begin{aligned}
AM &= \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} 2 & -11 \\ 2 & 12 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix} \end{bmatrix}
\end{aligned}
$$

Multiplication of Block Structured Matrices

$$
\begin{aligned}
AM &= \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} 2 & -11 \\ 2 & 12 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 4 & -12 \\ 6 & 15 \\ 1 & -2 \\ 4 & 1 \\ -4 & -1 \end{bmatrix}.
\end{aligned}
$$

Solving Systems of Linear equations

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
Find all solutions $x \in \mathbb{R}^n$ to the system $Ax = b$.

.

Solving Systems of Linear equations

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
Find all solutions $x \in \mathbb{R}^n$ to the system $Ax = b$.

The set of solutions is either

.

Solving Systems of Linear equations

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
Find all solutions $x \in \mathbb{R}^n$ to the system $Ax = b$.

The set of solutions is either empty,

.

Solving Systems of Linear equations

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
Find all solutions $x \in \mathbb{R}^n$ to the system $Ax = b$.

The set of solutions is either empty, a single point,

.

Solving Systems of Linear equations

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
Find all solutions $x \in \mathbb{R}^n$ to the system $Ax = b$.

The set of solutions is either empty, a single point, or an infinite set.

Solving Systems of Linear equations

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
Find all solutions $x \in \mathbb{R}^n$ to the system $Ax = b$.

The set of solutions is either empty, a single point, or an infinite set.

If a solution $x_0 \in \mathbb{R}^n$ exists, then the set of solutions is given by

$$x_0 \;+\; \mathrm{Nul}(A) \;.$$

Gaussian Elimination and the 3 Elementary Row Operations

We solve the system $Ax = b$ by transforming the augmented matrix

$$[\,A \,|\, b]$$

into upper echelon form using the three elementary row operations.

Gaussian Elimination and the 3 Elementary Row Operations

We solve the system $Ax = b$ by transforming the augmented matrix

$$[\,A \,|\, b]$$

into upper echelon form using the three elementary row operations. This process is called *Gaussian elimination*.

Gaussian Elimination and the 3 Elementary Row Operations

We solve the system $Ax = b$ by transforming the augmented matrix

$$[\,A \mid b]$$

into upper echelon form using the three elementary row operations.
This process is called *Gaussian elimination*.
The three elementary row operations.

Gaussian Elimination and the 3 Elementary Row Operations

We solve the system $Ax = b$ by transforming the augmented matrix

$$[\,A\,|\,b]$$

into upper echelon form using the three elementary row operations.
This process is called *Gaussian elimination*.
The three elementary row operations.

1. Interchange any two rows.

Gaussian Elimination and the 3 Elementary Row Operations

We solve the system $Ax = b$ by transforming the augmented matrix

$$[\,A \mid b]$$

into upper echelon form using the three elementary row operations.
This process is called *Gaussian elimination*.
The three elementary row operations.

1. Interchange any two rows.
2. Multiply any row by a non-zero constant.

Gaussian Elimination and the 3 Elementary Row Operations

We solve the system $Ax = b$ by transforming the augmented matrix

$$[\,A \mid b]$$

into upper echelon form using the three elementary row operations.
This process is called *Gaussian elimination*.
The three elementary row operations.

1. Interchange any two rows.
2. Multiply any row by a non-zero constant.
3. Replace any row by itself plus a multiple of any *other* row.

Gaussian Elimination and the 3 Elementary Row Operations

We solve the system $Ax = b$ by transforming the augmented matrix

$$[\,A \mid b]$$

into upper echelon form using the three elementary row operations.
This process is called *Gaussian elimination*.
The three elementary row operations.

1. Interchange any two rows.
2. Multiply any row by a non-zero constant.
3. Replace any row by itself plus a multiple of any *other* row.

These elementary row operations can be interpreted as multiplying the augmented matrix on the left by a special matrix.

Exchange and Permutation Matrices

An exchange matrix is given by permuting any two columns of the identity.

Exchange and Permutation Matrices

An exchange matrix is given by permuting any two columns of the identity.

Multiplying any $4 \times n$ matrix on the left by the exchange matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

will exchange the second and fourth rows of the matrix.

Exchange and Permutation Matrices

An exchange matrix is given by permuting any two columns of the identity.

Multiplying any $4 \times n$ matrix on the left by the exchange matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

will exchange the second and fourth rows of the matrix.

(multiplication of a $m \times 4$ matrix on the right by this exchanges the second and fourth columns.)

Exchange and Permutation Matrices

An exchange matrix is given by permuting any two columns of the identity.

Multiplying any $4 \times n$ matrix on the left by the exchange matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

will exchange the second and fourth rows of the matrix.

(multiplication of a $m \times 4$ matrix on the right by this exchanges the second and fourth columns.)
A permutation matrix is obtained by permuting the columns of the identity matrix.

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$.

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$.

Left Multiplication of A:

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$.

Left Multiplication of A:
When multiplying A on the left by an $m \times m$ matrix M, it is often useful to think of this as an action on the rows of A.

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$.

Left Multiplication of A:
When multiplying A on the left by an $m \times m$ matrix M, it is often useful to think of this as an action on the rows of A.

For example, left multiplication by a permutation matrix permutes the rows of the matrix.

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m\times n} \in \mathbb{R}^{m\times n}$.

Left Multiplication of A:
When multiplying A on the left by an $m \times m$ matrix M, it is often useful to think of this as an action on the rows of A.

For example, left multiplication by a permutation matrix permutes the rows of the matrix.

However, mechanically, left multiplication corresponds to matrix vector multiplication on the columns.

$$MA \;=\; [Ma_{\bullet 1} \; Ma_{\bullet 2} \; \cdots \; Ma_{\bullet n}]$$

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$.

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$.

Right Multiplication of A:

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$.

Right Multiplication of A:
When multiplying A on the right by an $n \times n$ matrix N, it is often useful to think of this as an action on the columns of A.

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m\times n} \in \mathbb{R}^{m\times n}$.

Right Multiplication of A:
When multiplying A on the right by an $n \times n$ matrix N, it is often useful to think of this as an action on the columns of A.

For example, right multiplication by a permutation matrix permutes the columns of the matrix.

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$.

Right Multiplication of A:
When multiplying A on the right by an $n \times n$ matrix N, it is often useful to think of this as an action on the columns of A.

For example, right multiplication by a permutation matrix permutes the columns of the matrix.

However, mechanically, right multiplication corresponds to left matrix vector multiplication on the rows.

Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$.

Right Multiplication of A:
When multiplying A on the right by an $n \times n$ matrix N, it is often useful to think of this as an action on the columns of A.

For example, right multiplication by a permutation matrix permutes the columns of the matrix.

However, mechanically, right multiplication corresponds to left matrix vector multiplication on the rows.

$$AN = \begin{bmatrix} a_{1\bullet}N \\ a_{2\bullet}N \\ \vdots \\ a_{m\bullet}N \end{bmatrix}$$

Gaussian Elimination Matrices

The key step in Gaussian elimination is to transform a vector of the form

$$\begin{bmatrix} a \\ \alpha \\ b \end{bmatrix},$$

where $a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$, into one of the form

$$\begin{bmatrix} a \\ \alpha \\ 0 \end{bmatrix}.$$

Gaussian Elimination Matrices

The key step in Gaussian elimination is to transform a vector of the form

$$\begin{bmatrix} a \\ \alpha \\ b \end{bmatrix},$$

where $a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$, into one of the form

$$\begin{bmatrix} a \\ \alpha \\ 0 \end{bmatrix}.$$

This can be accomplished by left matrix multiplication as follows.

Gaussian Elimination Matrices

$a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

Gaussian Elimination Matrices

$a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

Gaussian Elimination Matrices

$a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} a \\ \\ \\ \end{bmatrix}.$$

Gaussian Elimination Matrices

$a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} a \\ \alpha \\ \\ \end{bmatrix}.$$

Gaussian Elimination Matrices

$a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} a \\ \alpha \\ 0 \end{bmatrix}.$$

Gaussian Elimination Matrices

The matrix

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}$$

is called a Gaussian elimination matrix.

Gaussian Elimination Matrices

The matrix

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}$$

is called a Gaussian elimination matrix.

This matrix is invertible with inverse

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}.$$

Gaussian Elimination Matrices

The matrix

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}$$

is called a Gaussian elimination matrix.
This matrix is invertible with inverse

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}.$$

Note that a Gaussian elimination matrix and its inverse are both lower triangular matrices.

Matrix Sub-Algebras

Lower (upper) triangular matrices in $\mathbb{R}^{n \times n}$ are said to form a *sub-algebra* of $\mathbb{R}^{n \times n}$.

Matrix Sub-Algebras

Lower (upper) triangular matrices in $\mathbb{R}^{n\times n}$ are said to form a *sub-algebra* of $\mathbb{R}^{n\times n}$.

A subset S of $\mathbb{R}^{n\times n}$ is said to be a sub-algebra of $\mathbb{R}^{n\times n}$ if

Matrix Sub-Algebras

Lower (upper) triangular matrices in $\mathbb{R}^{n\times n}$ are said to form a *sub-algebra* of $\mathbb{R}^{n\times n}$.

A subset S of $\mathbb{R}^{n\times n}$ is said to be a sub-algebra of $\mathbb{R}^{n\times n}$ if

- S is a subspace of $\mathbb{R}^{n\times n}$,

Matrix Sub-Algebras

Lower (upper) triangular matrices in $\mathbb{R}^{n\times n}$ are said to form a *sub-algebra* of $\mathbb{R}^{n\times n}$.

A subset S of $\mathbb{R}^{n\times n}$ is said to be a sub-algebra of $\mathbb{R}^{n\times n}$ if

- S is a subspace of $\mathbb{R}^{n\times n}$,
- S is closed wrt matrix multiplication, and

Matrix Sub-Algebras

Lower (upper) triangular matrices in $\mathbb{R}^{n\times n}$ are said to form a *sub-algebra* of $\mathbb{R}^{n\times n}$.

A subset S of $\mathbb{R}^{n\times n}$ is said to be a sub-algebra of $\mathbb{R}^{n\times n}$ if

- S is a subspace of $\mathbb{R}^{n\times n}$,
- S is closed wrt matrix multiplication, and
- if $M \in S$ is invertible, then $M^{-1} \in S$.

Gaussian Elimination in Practice

Transformation to echelon
(upper triangular) form.

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}$$

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \\ & & \\ & & \end{bmatrix}$$

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ & & \\ & & \end{bmatrix}$$

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & & \\ & & \end{bmatrix}$$

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & \\ & & \end{bmatrix}$$

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ & & \end{bmatrix}$$

Gaussian Elimination in Practice

Transformation to echelon (upper triangular) form.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$

Eliminate the first column with a Gaussian elimination matrix.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix}$$

Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix}$$

Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}$$

Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & & \\ 0 & & \end{bmatrix}$$

Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \\ 0 & & \\ 0 & & \end{bmatrix}$$

Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & & \\ 0 & & \end{bmatrix}$$

Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & \\ 0 & & \end{bmatrix}$$

Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & & \end{bmatrix}$$

Gaussian Elimination in Practice

Now do Gaussian eliminiation on the second column.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 7 \end{bmatrix}$$

Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \\ \end{bmatrix}.$$

Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix}.$$

Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

What is the inverse of this matrix?

Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

What is the inverse of this matrix?

$$\begin{bmatrix} I_{k\times k} & a & 0 \\ 0 & \alpha & 0 \\ 0 & b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}.$$