Gaussian Elimination Matrices

Gaussian elimination transforms vectors of the form

\[
\begin{bmatrix}
a \\
\alpha \\
b
\end{bmatrix},
\]

where \(a \in \mathbb{R}^k\), \(0 \neq \alpha \in \mathbb{R}\), and \(b \in \mathbb{R}^{n-k-1}\), to those of the form

\[
\begin{bmatrix}
a \\
\alpha \\
0
\end{bmatrix}.
\]

This is accomplished by left matrix multiplication as follows:

\[
\begin{bmatrix}
l_{k \times k} & 0 & 0 \\
0 & 1 & 0 \\
0 & -\alpha^{-1}b & l_{(n-k-1) \times (n-k-1)}
\end{bmatrix}
\begin{bmatrix}
a \\
\alpha \\
b
\end{bmatrix} =
\begin{bmatrix}
a \\
\alpha \\
0
\end{bmatrix}.
\]
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\[
\begin{bmatrix}
  a \\
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  0
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This is accomplished by left matrix multiplication as follows:

\[
\begin{bmatrix}
  I_{k \times k} & 0 & 0 \\
  0 & 1 & 0 \\
  0 & -\alpha^{-1}b & I_{(n-k-1) \times (n-k-1)}
\end{bmatrix}
\begin{bmatrix}
  a \\
  \alpha \\
  b
\end{bmatrix} =
\begin{bmatrix}
  a \\
  \alpha \\
  0
\end{bmatrix}.
\]

The matrix on the left is called a Gaussian elimination matrix.
Gaussian Elimination Matrices

The matrix

\[
\begin{bmatrix}
I_{k \times k} & 0 & 0 \\
0 & 1 & 0 \\
0 & -\alpha^{-1}b & I_{(n-k-1) \times (n-k-1)}
\end{bmatrix}
\]

has ones on the diagonal and so is invertible. Indeed,

\[
\begin{bmatrix}
I_{k \times k} & 0 & 0 \\
0 & 1 & 0 \\
0 & -\alpha^{-1}b & I_{(n-k-1) \times (n-k-1)}
\end{bmatrix}^{-1} = \begin{bmatrix}
I_{k \times k} & 0 & 0 \\
0 & 1 & 0 \\
0 & \alpha^{-1}b & I_{(n-k-1) \times (n-k-1)}
\end{bmatrix}.
\]

Also note that

\[
\begin{bmatrix}
I_{k \times k} & 0 & 0 \\
0 & 1 & 0 \\
0 & -\alpha^{-1}b & I_{(n-k-1) \times (n-k-1)}
\end{bmatrix} \begin{bmatrix}
x \\
0 \\
y
\end{bmatrix} = \begin{bmatrix}
x \\
0 \\
y
\end{bmatrix}.
\]
LU Factorization

Suppose

\[ A = \begin{bmatrix} a_1 & v_1^T \\ u_1 & \tilde{A}_1 \end{bmatrix} \in \mathbb{C}^{n \times m}, \]

with \( 0 \neq a_1 \in \mathbb{C}, \ u_1 \in \mathbb{C}^{m-1}, \ v_1 \in \mathbb{C}^{n-1}, \) and \( \tilde{A}_1 \in \mathbb{C}^{(m-1) \times (n-1)} \).
LU Factorization

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Then

\[
\begin{bmatrix}
1 & 0 \\
-\frac{u_1}{a_1} & l
\end{bmatrix} \begin{bmatrix}
a_1 & v_1^T \\
u_1 & \tilde{A}_1
\end{bmatrix} \in \mathbb{C}^{n \times m} = \begin{bmatrix}
a_1 & v_1^T \\
0 & A_1
\end{bmatrix}, \quad (*)
\]

where \(A_1 = \tilde{A}_1 - u_1 v_1^T / a_1.\)
LU Factorization

Suppose

\[ A = \begin{bmatrix} a_1 & v_1^T \\ u_1 & \tilde{A}_1 \end{bmatrix} \in \mathbb{C}^{n \times m}, \]

with \( 0 \neq a_1 \in \mathbb{C} \), \( u_1 \in \mathbb{C}^{m-1} \), \( v_1 \in \mathbb{C}^{n-1} \), and \( \tilde{A}_1 \in \mathbb{C}^{(m-1) \times (n-1)} \).

Then

\[ \begin{bmatrix} 1 & 0 \\ -\frac{u_1}{a_1} & l \end{bmatrix} \begin{bmatrix} a_1 & v_1^T \\ u_1 & \tilde{A}_1 \end{bmatrix} \in \mathbb{C}^{n \times m} = \begin{bmatrix} a_1 & v_1^T \\ 0 & \tilde{A}_1 \end{bmatrix}, \tag{*} \]

where \( A_1 = \tilde{A}_1 - u_1 v_1^T / a_1 \).

Repeat \( m \) times to get \( L_{m-1}^{-1} \cdots L_2^{-1} L_1^{-1} A = U_{m-1} = U \) is upper triangular, so

\[ A = LU \]

where \( L \) is lower triangular with ones on the diagonal.
Suppose $M \in \mathbb{R}^{n \times n}$, symmetric and positive definite has LU factorization

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That is, $UL^{-T} = D$, where $D$ is diagonal.
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That is, $UL^{-T} = D$, where $D$ is diagonal.

Since $M$ is psd, $D$ has positive diagonal entries, so

$$M = LDL^T = \hat{L}\hat{L}^T$$

where $\hat{L} = LD^{1/2}$. 

This is called the Cholesky Factorization of $M$. 
Cholesky Factorization

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Householder Reflections

Given \( w \in \mathbb{R}^n \) we can associate the matrix

\[
U = I - 2 \frac{ww^T}{w^Tw}
\]

which reflects \( \mathbb{R}^n \) across the hyperplane \( \text{Span}\{w\}^\perp \). The matrix \( U \) is called the Householder reflection across this hyperplane.
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Given a pair of vectors \( x \) and \( y \) with

\[
\|x\|_2 = \|y\|_2, \quad \text{and} \quad x \neq y,
\]

there is a Householder reflection such that \( y = Ux \):

\[
U = I - 2 \frac{(x - y)(x - y)^T}{(x - y)^T(x - y)}.
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Householder Reflections

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\]

Householder reflections are symmetric unitary transformations:

\[
U^{-1} = U^T = U.
\]
The QR Factorization

Given $A \in \mathbb{R}^{m \times n}$ write

$$A_0 = \begin{bmatrix} \alpha_0 & a_0^T \\ b_0 & A_0 \end{bmatrix} \quad \text{and} \quad \nu_0 = \left\| \begin{bmatrix} \alpha_0 \\ b_0 \end{bmatrix} \right\|_2.$$
The QR Factorization

Given \( A \in \mathbb{R}^{m \times n} \) write

\[
A_0 = \begin{bmatrix}
\alpha_0 & a_0^T \\
b_0 & A_0
\end{bmatrix}
\quad \text{and} \quad
\nu_0 = \left\| \begin{pmatrix} \alpha_0 \\ b_0 \end{pmatrix} \right\|_2.
\]

Set

\[
H_0 = I - 2 \frac{ww^T}{w^Tw} \quad \text{where} \quad w = \begin{pmatrix} \alpha_0 \\ b_0 \end{pmatrix} - \nu_0 e_1 = \begin{pmatrix} \alpha_0 - \nu_0 \\ b_0 \end{pmatrix}.
\]
The QR Factorization

Given $A \in \mathbb{R}^{m \times n}$ write

$$A_0 = \begin{bmatrix} \alpha_0 & a_0^T \\ b_0 & A_0 \end{bmatrix} \quad \text{and} \quad \nu_0 = \left\| \begin{pmatrix} \alpha_0 \\ b_0 \end{pmatrix} \right\|_2.$$ 

Set

$$H_0 = I - 2 \frac{ww^T}{w^T w} \quad \text{where} \quad w = \begin{pmatrix} \alpha_0 \\ b_0 \end{pmatrix} - \nu_0 e_1 = \begin{pmatrix} \alpha_0 - \nu_0 \\ b_0 \end{pmatrix}.$$ 

Then

$$H_0 A = \begin{bmatrix} \nu_0 & a_1^T \\ 0 & A_1 \end{bmatrix}.$$
QR Factorization

\[ H_0A = \begin{bmatrix} \nu_0 & a_1^T \\ 0 & A_1 \end{bmatrix}. \]
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Repeat to get

\[ Q^T A = H_{n-1} H_{n-2} \ldots H_0 A = R, \]

where \( R \) is upper triangular and \( Q \) is unitary.
QR Factorization

\[ H_0 A = \begin{bmatrix} \nu_0 & a_1^T \\ 0 & A_1 \end{bmatrix}. \]

Repeat to get

\[ Q^T A = H_{n-1} H_{n-2} \cdots H_0 A = R, \]

where \( R \) is upper triangular and \( Q \) is unitary.

The \( A = QR \) is called the QR factorization of \( A \).
Orthogonal Projections

Suppose $A \in \mathbb{R}^{m \times n}$ with $m > n$, then $A =$
Suppose \( A \in \mathbb{R}^{m \times n} \) with \( m > n \), then \( A = \) 

The QR factorization of \( A \) looks like 

\[
A = [Q_1, \ Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix} = Q_1R
\]

where the columns of \( Q_1 \) and \( Q_2 \) form an orthonormal basis for \( \mathbb{R}^m \).
Suppose $A \in \mathbb{R}^{m \times n}$ with $m > n$, then $A = \mathbb{R}^m$

The QR factorization of $A$ looks like

$$A = [Q_1, \ Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix} = Q_1 R$$

where the columns of $Q_1$ and $Q_2$ for an orthonormal basis for $\mathbb{R}^m$. The columns of $Q_1$ form an orthonormal basis for the range of $A$ with

$$Q_1 Q_1^T = \text{the orthogonal projector onto Ran}(A)$$

and

$$I - Q_1 Q_1^T = Q_2 Q_2^T = \text{the orthogonal projector onto Ran}(A)^\perp$$
Similarly, if \( A \in \mathbb{R}^{m \times n} \) with \( m < n \), then \( A^T = \)
Orthogonal Projections

Similarly, if $A \in \mathbb{R}^{m \times n}$ with $m < n$, then $A^T = \text{something}$

The QR factorization of $A^T$ looks like

$$A^T = [Q_1, Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix} = Q_1 R$$

where the columns of $Q_1$ and $Q_2$ for an orthonormal basis for $\mathbb{R}^m$. 
Similarly, if $A \in \mathbb{R}^{m \times n}$ with $m < n$, then $A^T = \begin{bmatrix} \end{bmatrix}$

The QR factorization of $A^T$ looks like

$$A^T = [Q_1, \ Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix} = Q_1 R$$

where the columns of $Q_1$ and $Q_2$ for an orthonormal basis for $\mathbb{R}^m$. The columns of $Q_1$ form and orthonormal basis for the range of $A^T$ with

$$Q_1 Q_1^T = \text{the orthogonal projector onto } \text{Ran}(A^T)$$

and

$$I - Q_1 Q_1^T = Q_2 Q_2^T = \text{the orthogonal projector onto } \text{Ran}(A^T)^\perp = \text{Nul}(A)$$