Important Matrix Factorizations

LU Choleski QR

LU Factorization

Choleski Factorization

The QR Factorization

LU Factorization: Gaussian Elimination Matrices

Gaussian elimination transforms vectors of the form

$$\begin{bmatrix} a \\ \alpha \\ b \end{bmatrix},$$

where $a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$, to those of the form

$$\begin{bmatrix} a \\ \alpha \\ 0 \end{bmatrix}.$$

This is accomplished by left matrix multiplication as follows:

$$\begin{bmatrix} I_{k \times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1} b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} a \\ \alpha \\ 0 \end{bmatrix}.$$

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The matrix on the left is called a Gaussian elimination matrix.

Gaussian Elimination Matrices

The matrix

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}$$

has ones on the diagonal and so is invertible. Indeed,

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}^{-1} = \begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}.$$

Also note that

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} x \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ y \end{bmatrix}.$$

LU Factorization

Suppose

$$A = \begin{bmatrix} a_1 & v_1^T \\ u_1 & \widetilde{A}_1 \end{bmatrix} \in \mathbb{C}^{n\times m},$$

with $0 \neq a_1 \in \mathbb{C}$, $u_1 \in \mathbb{C}^{m-1}$, $v_1 \in \mathbb{C}^{n-1}$, and $\widetilde{A}_1 \in \mathbb{C}^{(m-1)\times(n-1)}$.

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Then

$$\begin{bmatrix} 1 & 0 \\ -\frac{u_1}{a_1} & I \end{bmatrix} \begin{bmatrix} a_1 & v_1^T \\ u_1 & \widetilde{A}_1 \end{bmatrix} \in \mathbb{C}^{n\times m} = \begin{bmatrix} a_1 & v_1^T \\ 0 & A_1 \end{bmatrix}, \qquad (*)$$

where $A_1 = \widetilde{A}_1 - u_1 v_1^T / a_1$.

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where $A_1 = \widetilde{A}_1 - u_1 v_1^T / a_1$.
Repeat m times to get $L_{\tilde{m}-1}^{-1} \cdots L_2^{-1} L_1^{-1} A = U_{\tilde{m}-1} = U$ is upper triangular, so

$$A = LU$$

where L is lower triangular with ones on the diagonal.

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This is called the Cholesky Factorization of M.

The QR Factorization: Householder Reflections

Given $w \in \mathbb{R}^n$ we can associate the matrix

$$U = I - 2\frac{ww^T}{w^Tw}$$

which reflects $\mathbb{R}^n$ across the hyperplane $\mathrm{Span}\{w\}^{\perp}$. The matrix U is call the Householder reflection across this hyperplane.

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Given a pair of vectors x and y with

$$\|x\|_2 = \|y\|_2, \quad \text{and} \quad x \neq y,$$

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Householder reflections are symmetric unitary tranformations: $U^{-1} = U^T = U$.

The QR Factorization

Given $A \in \mathbb{R}^{m \times n}$ write

$$A_0 = \begin{bmatrix} \alpha_0 & a_0^T \\ b_0 & A_0 \end{bmatrix} \qquad \text{and} \qquad \nu_0 = \left\| \begin{pmatrix} \alpha_0 \\ b_0 \end{pmatrix} \right\|_2 .$$

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Set

$$H_0 = I - 2\frac{ww^T}{w^T w} \qquad \text{where} \qquad w = \begin{pmatrix} \alpha_0 \\ b_0 \end{pmatrix} - \nu_0 e_1 = \begin{pmatrix} \alpha_0 - \nu_0 \\ b_0 \end{pmatrix} .$$

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Then

$$H_0 A = \begin{bmatrix} \nu_0 & a_1^T \\ 0 & A_1 \end{bmatrix} .$$

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Repeat to get

$$Q^T A = H_{n-1} H_{n-2} \ldots H_0 A = R,$$

where R is upper triangular and Q is unitary.

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where R is upper triangular and Q is unitary.

The $A = QR$ is called the QR factorization of A.

Orthogonal Projections

Suppose $A \in \mathbb{R}^{m \times n}$ with $m > n$, then $A = \boxed{\phantom{\begin{matrix} \\ \\ \\ \\ \end{matrix}}}$

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$$Q_1 Q_1^T \ = \ \text{the orthogonal projector onto } \operatorname{Ran}(A)$$

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The columns of Q_1 form and orthonormal basis for the range of A^T with

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