Math 408 Homework Set 7

(1) Find the global minimizers and maximizers, if they exist, for the following functions.
  (a) $f(x) = x_1^2 - 4x_1 + 2x_2^2 + 7$
  (b) $f(x) = e^{-\|x\|^2}$
  (c) $f(x) = x_1^2 - 2x_1x_2 + \frac{1}{3}x_2^3 - 8x_2$
  (d) $f(x) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$
  (e) $f(x) = x_1^4 + 16x_1x_2 + x_2^4$
  (f) $f(x) = (1 - x_1)^2 + \sum_{j=1}^{n-1} 10^j (x_j - x_{j+1}^2)^2$ (The Rosenbrock function)

(2) Locate all of the KKT points for the following problems. Can you show that these points are local solutions? Global solutions?
  (a)
$$\begin{array}{ll} \text{minimize} & e^{(x_1 - x_2)} \\ \text{subject to} & e^{x_1} + e^{x_2} \le 20 \\ & 0 \le x_1 \end{array}$$
  (b)
$$\begin{array}{ll} \text{minimize} & e^{(-x_1 + x_2)} \\ \text{subject to} & e^{x_1} + e^{x_2} \le 20 \\ & 0 \le x_1 \end{array}$$
  (c)
$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 - 4x_1 - 4x_2 \\ \text{subject to} & x_1^2 \le x_2 \\ & x_1 + x_2 \le 2 \end{array}$$
  (d)
$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x\|^2 \\ \text{subject to} & Ax = b \end{array}$$
  where $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ satisfies $\mathrm{Nul}\,(A^T) = \{0\}$.

(3) Show that the set
$$\Omega := \{x \in \mathbb{R}^2 | - x_1^3 \le x_2 \le x_1^3\}$$
is not regular at the origin. Graph the set Ω.

(4) Construct an example of a constraint region of the form (**??**) at which the MFCQ is satisfied, but the LI condition is not satisfied.

(5) Suppose $\Omega = \{x\,;\, Ax \le b,\ Ex = h\}$ where $A \in \mathbb{R}^{m\times}$, $E \in \mathbb{R}^{k \times n}$, $b \in \mathbb{R}^m$, and $h \in \mathbb{R}^k$.
  (a) Given $x \in \Omega$, show that
$$T_\Omega(x) = \{d \,:\, A_{i\cdot}d \le 0 \text{ for } i \in I(x),\ Ed = 0\},$$
  where $A_{i\cdot}$ denotes the ith row of the matrix A and $I(x) = \{i\ A_{i\cdot}x = b_i\}$.
  (b) Given $x \in \Omega$, show that every $d \in T_\Omega(x)$ is a feasible direction for Ω at x.
  (c) Note that parts (a) and (b) above show that
$$T_\Omega(x) = \bigcup_{\lambda > 0} \lambda(\Omega - x)$$
  whenever Ω is a convex polyhedral set. Why?

(6) Show that each of the following functions is convex or strictly convex.
  (a) $f(x, y) = 5x^2 + 2xy + y^2 - x + 2y + 3$
  (b) $f(x, y) = \begin{cases} (x + 2y + 1)^8 - \log((xy)^2), & \text{if } 0 < x,\ 0 < y, \\ +\infty, & \text{otherwise.} \end{cases}$
  (c) $f(x, y) = 4e^{3x - y} + 5e^{x^2 + y^2}$
  (d) $f(x, y) = \begin{cases} x + \frac{2}{x} + 2y + \frac{4}{y}, & \text{if } 0 < x,\ 0 < y, \\ +\infty, & \text{otherwise.} \end{cases}$

(7) Consider the global minimizers of the functions given in the previous problem if they exist.
  (a) Compute the unique global minimizer.

(b) Show that that global minimizer is obtained by solving the equation $4x(x + \sqrt{2x} + 1)^7 = 1$ for $x > 0$, then setting $y = \sqrt{x/2}$.
(c) Show that the unique global solution is given by numerically solving the equation $5y \exp(10(y^2 + 1)) = 2$ for y then set $x = -3y$.
(d) Compute the unique global minimizer.

(8) Let $Q \in \mathcal{S}^n_{++}$ and $c \in \mathbb{R}^n$. By making explicit use of Q^{-1}, compute the Lagrangian dual to the convex quadratic program

$$\mathcal{Q} \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Qx + c^T x \\ \text{subject to} & Ax \leq b,\ 0 \leq x\ . \end{array}$$

(9) Consider the functions

$$f(x) = \frac{1}{2}x^T Qx - c^T x$$

and

$$f_t(x) = \frac{1}{2}x^T Qx - c^T x + t\phi(x),$$

where $t > 0$, $Q \in \mathbb{S}^n_+$, $c \in \mathbb{R}^n$, and $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\phi(x) = \begin{cases} -\sum_{i=1}^n \ln x_i & \text{, if } x_i > 0,\ i = 1, 2, \ldots, n, \\ +\infty & \text{, otherwise.} \end{cases}$$

(a) Show that ϕ is a convex function.
(b) Show that both f and f_t are convex functions.
(c) Show that the solution to the problem $\min f_t(x)$ always exists and is unique.
(d) Let $\{t_i\}$ be a decreasing sequence of positive real scalars with $t_i \downarrow 0$, and let x^i be the solution to the problem $\min f_{t_i}(x)$. Show that if the sequence $\{x^i\}$ has a cluster point $\bar{x}$, then $\bar{x}$ must be a solution to the problem $\min\{f(x) : 0 \leq x\}$.
Hint: Use the KKT conditions for the QP $\min\{f(x) : 0 \leq x\}$.