Math 408 Homework Set 6

(1) Show that the functions

$$f(x_1, x_2) = x_1^2 + x_2^3, \quad \text{and} \quad g(x_1, x_2) = x_1^2 + x_2^4$$

both have a critical point at $(x_1, x_2) = (0, 0)$ and that their associated Hessians are positive semi-definite. Then show that $(0, 0)$ is a local (global) minimizer for g and not for f.

(2) Find the local minimizers and maximizers for the following functions if they exist:
   (a) $f(x) = x^2 + \cos x$
   (b) $f(x_1, x_2) = x_1^2 - 4x_1 + 2x_2^2 + 7$
   (c) $f(x_1, x_2) = e^{-(x_1^2 + x_2^2)}$
   (d) $f(x_1, x_2, x_3) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$

(3) Compute the directional derivative for each of the following functions at the origin.
   (a) $f(x) = \max\{0, x\}$
   (b) $f(x) = \max\{-x,\ 2x\}$
   (c) $f(x_1, x_2) = |x_1| - |x_2|$

(4) Show that the function $f(x) := \frac{1}{2}(\max\{0,\ x\})^2$ is differentiable at the origin and give its derivative.

(5) Let $C \subset \mathbb{R}^n$ and $x \in C$ and recall the definition of the tangent cone to C at x:

$$T_C(x) := \left\{ u \;\middle|\; \exists\ \{x^\nu\} \subset C,\ x^\nu \to x,\ t_\nu \downarrow 0, \ \text{ with } \ t_\nu^{-1}(x^\nu - x) \to u \right\}.$$

   (a) Let $\mathbb{B}_2 = \{u \;|\|u\|_2 \le 1\}$. Show that for all $u \in \mathbb{B}_2$ with $\|u\|_2 = 1$,

$$T_{\mathbb{B}_2}(u) = \left\{ v \;\middle|\; u^T v \le 0 \right\}.$$

   (b) Consider the continuous function

$$f(x) := \begin{cases} -\sqrt{\|x\|_2^2 - 1} & \text{, if } \|x\|_2 \ge 1, \text{ and} \\ 0 & \text{, if } \|x\|_2 < 1. \end{cases}$$

   Obviously, $\mathbb{B}_2 = \operatorname{argmin}\{f(x) \;|x \in \mathbb{B}_2\}$, since f is identically zero on $\mathbb{B}_2$. Let $\|u\|_2 = 1 = \|v\|_2$ with $u^T v = 0$ so that $v \in T_{\mathbb{B}_2}(u)$. Show that $f'(u; v)$ exists with $f'(u; v) = -1$, where

$$f'(u; v) := \lim_{t \downarrow 0} \frac{f(u + tv) - f(u)}{t}.$$

   (c) Let $h : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and let $S \subset \mathbb{R}^n$. Show that if $\bar{x} \in \operatorname{argmin}\{h(x) \;|x \in S\}$, then $h'(\bar{x}; d) \ge 0$ for all $d \in T_S(\bar{x})$. Does this result contradict your finding in part (5b)? If not, why not?

(6) Show that the representation of the set $\Omega := \{x \in \mathbb{R}^2 | -x_1^3 \le x_2 \le x_1^3\}$ is not regular at the origin. Can you suggest an alternative representation that is regular at the origin?

(7) Let Ω be given the representation $\Omega := \{x \in \mathbb{R}^2 \;|x_2 \le 0,\ -x_2 \le 0\}$ and consider the optimization problem $\min\{x_1^2 \;|x \in \Omega\}$. Show that the unique global minimzer of this problem satisfies the MFCQ but not the LICQ. Also, compute the set of KKT multipliers for this global solution.

(8) Locate all of the KKT points for the following problems. Can you show that these points are local solutions? Global solutions?

(a)

$$\begin{array}{ll} \text{minimize} & e^{(x_1-x_2)} \\ \text{subject to} & e^{x_1}+e^{x_2}\le 20 \\ & 0\le x_1 \end{array}$$

(b)

$$\begin{array}{ll} \text{minimize} & e^{(-x_1+x_2)} \\ \text{subject to} & e^{x_1}+e^{x_2}\le 20 \\ & 0\le x_1 \end{array}$$

(c)

$$\begin{array}{ll} \text{minimize} & x_1^2+x_2^2-4x_1-4x_2 \\ \text{subject to} & x_1^2\le x_2 \\ & x_1+x_2\le 2 \end{array}$$

(d)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x\|^2 \\ \text{subject to} & Ax=b \end{array}$$

where $b\in\mathbb{R}^m$ and $A\in\mathbb{R}^{m\times n}$ satisfies $\mathrm{Nul}\,(A^T)=\{0\}$.

(9) Suppose $\Omega=\{x\,;\,Ax\le b,\ Ex=h\}$ where $A\in\mathbb{R}^{m\times}$, $E\in\mathbb{R}^{k\times n}$, $b\in\mathbb{R}^m$, and $h\in\mathbb{R}^k$.

(a) Given $x\in\Omega$, show that

$$T_\Omega(x)=\{d\,:\,A_{i\cdot}d\le 0 \text{ for } i\in I(x),\ Ed=0\},$$

where $A_{i\cdot}$ denotes the ith row of the matrix A and $I(x)=\{i\ A_{i\cdot}x=b_i\}$.

(b) Given $x\in\Omega$, show that every $d\in T_\Omega(x)$ is a feasible direction for Ω at x.

(c) Note that parts (a) and (b) above show that

$$T_\Omega(x)=\bigcup_{\lambda>0}\lambda(\Omega-x)$$

whenever Ω is a convex polyhedral set. Why?