Math 408 Homework Set 5

(1) (Orthogonal Projections) A matrix $P \in \mathbb{R}^{n\times n}$ is said to be a projection if $P^2 = P$. It is said to be an orthogonal projection with respect to the inner product $\langle \cdot, \cdot \rangle$ if $P^2 = P$ and $\langle (I-P)x, Py \rangle = 0$ for all $x, y \in \mathbb{R}^n$.

(a) Show that for every $x \in \mathrm{Ran}\,(P)$, $Px = x$. For this reason we say that $P$ projects on the subspace $\mathrm{Ran}\,(P)$.

(b) Show that the projection $P$ is an orthogonal projection (wrt the usual inner product) if and only if $P^T = P$.

(c) If $P$ is an orthogonal projection, show that $Q = I - P$ is the orthogonal projection onto $\mathrm{Ran}\,(P)^{\perp}$.

(d) If $P$ is an orthogonal projection (wrt the usual inner product), show that

$$\|x\|_2^2 = \|Px\|_2^2 + \|(I-P)x\|_2^2 \quad \forall\, x \in \mathbb{R}^n\ .$$

(e) Let $A \in \mathbb{R}^{m\times n}$ be such that $\mathrm{Nul}\,(A) = \{0\}$. Show that $(A^TA)^{-1}$ exists and that the orthogonal projection onto $\mathrm{Ran}\,(A)$ is given by $P = A(A^TA)^{-1}A^T$.

(f) Let $A \in \mathbb{R}^{m\times n}$ be such that $\mathrm{Ran}\,(A) = \mathbb{R}^m$. Show that $(AA^T)^{-1}$ exists and that the orthogonal projection onto $\mathrm{Nul}\,(A)$ is given by $Q = I - A^T(AA^T)^{-1}A$.

(g) Let $A \in \mathbb{R}^{m\times n}$ be such that $\mathrm{Nul}\,(A) = \{0\}$, and let $P = A(A^TA)^{-1}A^T$. Show that

$$\frac{1}{2}\|(I-P)b\|_2^2 = \min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2.$$

(h) ** Let $A \in \mathbb{R}^{m\times n}$ be such that $\mathrm{Ran}\,(A) = \mathbb{R}^m$ and set $P = A^T(AA^T)^{-1}A$. Show that $Px^0 = A^T(AA^T)^{-1}b$ for every $x^0 \in \mathbb{R}^n$ satisfying $Ax^0 = b$ and that $\hat{x} := A^T(AA^T)^{-1}b$ is the unique solution to the problem

$$\min \frac{1}{2}\|x\|_2^2 \quad \text{subject to} \quad Ax = b\ .$$

(2) (Best approximation on the linear span of a finite collection of basis functions) Let $f_j : \mathbb{R} \to \mathbb{R}$ $j = 1, 2, \ldots, n$ be a collection of known functions and let $\mathcal{S} := \mathrm{Span}[f_1, f_2, \ldots, f_n]$ be the linear span of these functions. Given data $\{(x_i, y_i)\ |i = 1, 2, \ldots m\}$, one can ask which function in $\mathcal{S}$ "best" fits this data. If we measure "best" in the sense of the 2-norm, then the answer to this question is given by solving the problem

$$\min_{f\in\mathcal{S}} \frac{1}{2}\sum_{i=1}^{m}(y_i - f(x_i))^2\ .$$

Show that this problem can be recast as a linear least squares problem.
Hint: What if $f_j(x) = x^j$?