Math 408 Homework Set 4

(1) Let H, A, and b be as above and define $\ell : \mathbb{R}^k \to \mathbb{R}$ by

$$\ell(x) := \frac{1}{2}(Ax-b)^T Q(Ax-b) \ ,$$

and consider the optimization problem

$$\mathcal{Q}-\mathcal{LLS} \qquad \min_{x\in\mathbb{R}^n} \frac{1}{2}(Ax-b)^T Q(Ax-b).$$

- (a) Give necessary and sufficient conditions under which the optimization problem $\mathcal{Q}-\mathcal{LLS}$ has a global optimal solution.
- (b) If Q is positive definite, show that $\mathcal{Q}-\mathcal{LLS}$ is equivalent to a linear least squares problem.
- (c) Give a necessary and sufficient condition under which $\mathcal{Q}-\mathcal{LLS}$ has a unique global optimal solution.

(2) Determine whether the following matrices are positive definite, positive semi-definite, or neither by attempting to compute their Choleski factorizations.

$$(a)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (b)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$(c)\ H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad (d)\ H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} .$$

(3) Consider the linearly constrained quadratic optimization problem

$$\mathcal{Q}(H,g,A,b) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Hx + g^T x \\ \text{subject to} & Ax = b \ , \end{array}$$

where $H \in \mathbb{R}^{n\times n}$ is symmetric an positive definite and $A \in \mathbb{R}^{m\times n}$ has rank $(A) = m$.

- (a) Write necessary and sufficient optimality conditions for this problem at a pair $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ where $\bar{y}$ is an Lagrange multiplier vector.
- (b) Solve the problem $\mathcal{Q}(H,g,A,b)$ with

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix},\ g = (1,1,1)^T,\ b = (4,2)^T, \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

- (c) Solve the problem $\mathcal{Q}(H,g,A,b)$ with $g = 0$, $m = 1$, $b = \mu \in \mathbb{R}$, and $A = \mathbf{1}_n^T$, where $\mathbf{1}_n \in \mathbb{R}^n$ is the vector of all ones.
- (d) Show that the matrix $AH^{-1}A^T$ is invertible.
- (e) Show that $\bar{y} = -(AH^{-1}A^T)^{-1}(AH^{-1}g + b)$.
- (f) Show that $\bar{x} = -[H^{-1} - H^{-1}A^T(AH^{-1}A^T)^{-1}AH^{-1}]g + H^{-1}A^T(AH^{-1}A^T)^{-1}b$.
- (g) Show that

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix}^{-1} = \begin{bmatrix} [H^{-1} - H^{-1}A^T(AH^{-1}A^T)^{-1}AH^{-1}] & H^{-1}A^T(AH^{-1}A^T)^{-1} \\ (AH^{-1}A^T)^{-1}AH^{-1} & -(AH^{-1}A^T)^{-1} \end{bmatrix}.$$

(4) Prove Proposition 2.1 in Chapter 3.

(5) Prove Part (5) of Theorem 2.1 in Chapter 3 on the course notes.

(6) Use Lemma 3.1 Part (1) to inductively show that the only matrix in $\mathcal{S}^n_+$ having zero diagonal is the zero matrix.

(7) Let $A \in \mathbb{R}^{m\times n}$ have rank $m < n$ and let $H \in \mathcal{S}^n$ be positive definite on $\mathrm{Nul}\,(A)$, i.e., $u^T H u > 0,\ \forall\, u \in \mathrm{Nul}\,(A)$. These hypotheses imply that AA^T is invertible. Define $A^\dagger := A^T(AA^T)^{-1}$. In this context the matrix $A^\dagger$ is called the Moore-Penrose pseudo inverse of A.

(a) Show that the dimension of the $\mathrm{Nul}\,(A)$ is $n - m$.

(b) Let $U \in \mathbb{R}^{n\times(n-m)}$ be *any* matrix whose columns form an orthonormal basis for $\mathrm{Nul}\,(A)$ and show that $I - UU^T = A^\dagger A$.

(c) Show that

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix}^{-1} = \begin{pmatrix} U(U^T HU)^{-1}U^T & \left(I - U(U^T HU)^{-1}U^T H\right) A^\dagger \\ (A^\dagger)^T \left(I - HU(U^T HU)^{-1}U^T\right) & (A^\dagger)^T \left(HU(U^T HU)^{-1}U^T H - H\right) A^\dagger \end{pmatrix},$$

where $U \in \mathbb{R}^{n\times(n-p)}$ is any matrix whose columns form an orthonormal basis of $\ker A$.