Math 408 Homework Set 3

This homework set will focus on the optimization problem

$$\mathcal{Q} \qquad \min_{x\in\mathbb{R}^n} \frac{1}{2}x^T Hx + g^T x \ ,$$

where $H \in \mathbb{R}^{n\times n}$ is symmetric and $g \in \mathbb{R}^n$.

(1) Each of the following functions can be written in the form $f(x) = \frac{1}{2}x^T Hx + g^T x$ with H symmetric. For each of these functions what are H and g
  (a) $f(x) = x_1^2 - 4x_1 + 2x_2^2 + 7$
  (c) $f(x) = x_1^2 - 2x_1x_2 + \frac{1}{2}x_2^2 - 8x_2$
  (d) $f(x) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$
  (e) $f(x) = x_1^2 + 16x_1x_2 + 4x_2x_3 + x_2^2$

(2) Let $H \in \mathcal{S}^n_{++}$ and g, $\bar{x}$, $d \in \mathbb{R}^n$ with $d \neq 0$, and consider the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \frac{1}{2}x^T Hx + g^T x \quad\text{and}\quad \phi(t) := f(\bar{x} + td).$$

What is the solution to the problem $\min_{t\in\mathbb{R}} \phi(t)$?

(3) Let α, $\beta \in \mathbb{R}$ and define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(t : \alpha, \beta) := \frac{\alpha}{2}t^2 + \beta t$. Compute the values $\bar{\phi}(\alpha, \beta) := \min\{\phi(t : \alpha, \beta) \ |0 \le t \le 1\}$ and the set $\Gamma(\alpha, \beta) := \operatorname{argmin}\{\phi(t; \alpha, \beta) \ |0 \le t \le 1\}$ for all possible values of $(\alpha, \beta) \in \mathbb{R}^2$.

(4) For each of the matrices H and vectors g below determine the optimal value in $\mathcal{Q}$. If an optimal solution exists, compute the complete set of optimal solutions.
  (a)

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad\text{and}\quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

  (b)

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad\text{and}\quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

  (c)

$$H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad\text{and}\quad g = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

(5) Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $g \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad\text{and}\quad g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Does there exists a vector $u \in \mathbb{R}^3$ such that $f(tu) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct u.

(6) A mapping $\langle\cdot\,,\,\cdot\rangle : \mathbb{R}^n \to \mathbb{R}^n$ is said to be an inner product on $\mathbb{R}^n$ is for all $x, y, z \in \mathbb{R}^n$

(i)	$\langle x\,,\,x\rangle \geq 0$	Non-Negative
(ii)	$\langle x\,,\,x\rangle = 0 \Leftrightarrow x = 0$	Positive
(iii)	$\langle x+y\,,\,z\rangle = \langle x\,,\,z\rangle + \langle y\,,\,z\rangle$	Additive
(iv)	$\langle \alpha x\,,\,y\rangle = \alpha\langle x\,,\,y\rangle \;\forall\, \alpha \in \mathbb{R}$	Homogeneous
(v)	$\langle x\,,\,y\rangle = \langle y\,,\,x\rangle$	Symmetric

Two vectors $x, y \in \mathbb{R}^n$ are said to be orthogonal in the inner product $\langle \cdot\,,\,\cdot\rangle$ if $\langle x\,,\,y\rangle = 0$.
Unless otherwise specified, let $\langle x\,,\,y\rangle$ denote the usual Euclidean inner product:

$$\langle x\,,\,y\rangle = x^T y = \sum_{i=1}^{n} x_i y_i \;.$$

(a) Let $\langle x\,,\,y\rangle$ be the standard Euclidean inner product on $\mathbb{R}^n$. Given $A \in \mathbb{R}^{n\times n}$, show that $A = 0$ if and only if

$$\langle x\,,\,Ay\rangle = 0 \quad \forall\, x, y \in \mathbb{R}^n \;.$$

(b) Let $H \in \mathbb{R}^{n\times n}$ be symmetric and positive definite (i.e. $H = H^T$ and $x^T H x > 0 \;\forall\, x \in \mathbb{R}^n \setminus \{0\}$). Show that the relation

$$\langle x\,,\,y\rangle_H := x^T H y \quad \forall\, x, y \in \mathbb{R}^n$$

defines an inner product on $\mathbb{R}^n$.

(c) Every inner product defines a transformation on the space of linear operators called the *adjoint*. For the Euclidean inner product on $\mathbb{R}^n$, this is just the usual transpose. Given a linear transformation $M : \mathbb{R}^n \to \mathbb{R}^n$, the adjoint is defined by the relation

$$\langle y\,,\,Mx\rangle = \langle M^* y\,,\,x\rangle, \quad \text{for all } x, y, \in \mathbb{R}^n.$$

The inner product $\langle \cdot\,,\,\cdot\rangle_H$ defined above, also defines an adjoint mapping which we can denote by M^{T_H}. Show that

$$M^{T_H} = H^{-1} M^T H \;.$$

(d) The matrix $P \in \mathbb{R}^{n\times n}$ is said to a projection if $P^2 = P$. Clearly, if P is a projection, then so is $I - P$. The subspace $P\mathbb{R}^n = \mathrm{Ran}\,(P)$ is called the subspace that P projects onto. A projection is said to be orthogonal with respect to a given inner product $\langle \cdot\,,\,\cdot\rangle$ on $\mathbb{R}^n$ if and only if

$$\langle (I-P)x\,,\,Py\rangle = 0 \quad \forall\, x, y \in \mathbb{R}^n \;,$$

that is, the subspaces $\mathrm{Ran}\,(P)$ and $\mathrm{Ran}\,(I-P)$ are orthogonal in the inner product $\langle \cdot\,,\,\cdot\rangle$. Show that the projection P is orthogonal with respect to the inner product $\langle \cdot\,,\,\cdot\rangle_H$ (defined above), where $H \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, if and only if

$$P = H^{-1} P^T H \;.$$