Math 408 Homework Set 1

Linear Algebra Review Problems

(1) Consider the system

$$\begin{array}{rcrcrcl} 4x_1 & & & - & x_3 & = & 200 \\ 9x_1 & + & x_2 & - & x_3 & = & 200 \\ 7x_1 & - & x_2 & + & 2x_3 & = & 200\ . \end{array}$$

(a) Write the augmented matrix corresponding to this system.
(b) Reduce the augmented system in part (a) to echelon form.
(c) Describe the set of solutions to the given system.

(2) Represent the linear span of the four vectors

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 1 \\ 7 \\ 1 \end{bmatrix}, \quad \text{and} \quad x_4 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 5 \end{bmatrix},$$

as the range space of some matrix.

(3) Compute a basis for $\text{nul}\,(A^T)^\perp$ where A is given by

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 2 & 1 & 7 & 0 \\ 1 & -2 & 1 & 5 \end{bmatrix}.$$

(4) Find the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -4 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(5) Solve the following system of linear equations

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & & & = & 1 \\ -x_1 & - & 4x_2 & + & x_3 & = & 2 \\ & & 2x_2 & + & x_3 & = & 0. \end{array}$$

(6) Determine whether the following system of linear equations has a solution or not.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 0 \end{pmatrix}.$$

(7) Find a 2 by 2 square matrix B satisfying

$$A = B \cdot C,$$

where $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & -3 & 0 \\ 8 & 9 & 3 \end{pmatrix}$.

(8) Show that the Gaussian elimination matrix for the vector

$$v = \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

where the pivot $\alpha \in \mathbb{R}$ is non-zero, $a \in \mathbb{R}^k$, and $b \in \mathbb{R}^{(n-(k+1))}$ is non-singular by providing an expression for its inverse.

(9) What is the Gaussian elimination matrix for the vector

$$v = \begin{bmatrix} 2 \\ -10 \\ 16 \\ 2 \end{bmatrix}?$$

where the entry $x_2 = 2$ is the pivot? What is it if the pivot is $x_2 = -10$?

(10) Let $A \in \mathbb{R}^{k\times n}$ and $B \in \mathbb{R}^{k\times m}$ so that $A^T B \in \mathbb{R}^{n\times m}$. If $(A^T B)_{ij}$ is the ij^{th} element if $A^T B$, show that $(A^T B)_{ij} = a_i^T b_j$ where a_i and b_j are the i^{th} and j^{th} columns of A and B, respectively.

(11) Show that the product of two lower triangular $n \times n$ matrices is always a lower triangular matrix.

(12) Show that the inverse of a non-singular lower triangular matrix is always lower triangular.

(13) A *Housholder transformation* on $\mathbb{R}^n$ is any $n \times n$ matrix of the form

$$P = I - 2\frac{vv^T}{v^T v}$$

for some non-zero vector $v \in \mathbb{R}^n$. The Householder transformation is the reflection across the hyperplane $v^T x = 0$.

(a) Given any two vectors u and w in $\mathbb{R}^n$ such that $\|u\| = \|w\|$ and $u \neq w$, if P is the Householder transformation based on the vector $v := u - w$, show that $Pu = w$.

(b) If

$$u = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

explicitly construct the Householder transformation for which $Pu = w$.

(c) Show that every Householder transformation P satisfies $P^T = P$ and $P^2 = I$.

Multi-variable Calculus Review Problems

(1) Find the local and global minimizers and maximizers of the following functions.

(a) $f(x) = x^2 + 2x$

(b) $f(x) = x^2 e^{-x^2}$

(c) $f(x) = x^2 + \sin x$

(d) $f(x) = x^3 - x$

(2) Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be differentiable at a point $x \in \mathbb{R}^n$ if there is a vector $g \in \mathbb{R}^n$ such that

$$f(y) = f(x) + g^T(y - x) + o(\|y - x\|) \ .$$

The vector g is called the gradient of f at x and is denoted $g = \nabla f(x)$. Note that, when defined, the relation $x \mapsto \nabla f(x)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e. $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$. We say that f is continuously differentiable at $x \in \mathbb{R}^n$ if the mapping ∇f is continuous at x. When f is continuously differentiable at $x \in \mathbb{R}^n$, then $\nabla f(x)$ is easily computed as the vector of partial derivatives of f at x, i.e.

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}.$$

Compute the gradient of the following functions.

(a) $f(x) = x_1^3 + x_2^3 - 3x_1 - 15x_2 + 25 : \qquad f : \mathbb{R}^2 \to \mathbb{R}$

(b) $f(x) = x_1^2 + x_2^2 - \sin(x_1 x_2) \qquad f : \mathbb{R}^2 \to \mathbb{R}$

(c) $f(x) = \|x\|^2 = \sum_{j=1}^n x_j^2 : \qquad f : \mathbb{R}^n \to \mathbb{R}$

(d) $f(x) = e^{\|x\|^2}$

(e) $f(x) = x_1 x_2 x_3 \cdots x_n : \qquad f : \mathbb{R}^n \to \mathbb{R}$

(f) $f(x) = -\log(x_1 x_2 x_3 \cdots x_n)$ for $x_j > 0,\ j = 1, \ldots n$, and undefined otherwise: $\qquad f : \mathbb{R}^n \to \mathbb{R}$. Compute $\nabla f(x)$ for $x_j > 0,\ j = 1, \ldots n$.

(3) Let $\mathbb{R}^{n \times n}$ denote the set of real $n \times n$ square matrices A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be twice differentiable at a point $x \in \mathbb{R}^n$ if is differentiable at x and there is a matrix $H \in \mathbb{R}^{n \times n}$ such that

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T H (y - x) + o(\|y - x\|^2).$$

The matrix H is called the Hessian of f at x and is denoted $\nabla^2 f(x)$. Note that, when defined, the relation $x \mapsto \nabla^2 f(x)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^{n \times n}$, i.e. $\nabla^2 f : \mathbb{R}^n \to \mathbb{R}^n$. We say that f is twice continuously differentiable at $x \in \mathbb{R}^n$ if the mapping $\nabla^2 f$ is continuous at x. It can be shown that if f is twice continuously differentiable at a point $x \in \mathbb{R}^n$, then the matrix $\nabla^2 f(x)$ is symmetric, i.e. $\nabla^2 f(x) = \nabla^2 f(x)^T$, in which case $\nabla^2 f(x)$ is the matrix of second partial derivatives of f at x:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(x) \end{bmatrix}.$$

Compute the Hessian of the functions given in problem (2) above.

(4) Let $b \in \mathbb{R}^m$ and consider the matrix $A \in \mathbb{R}^{m \times n}$ given by

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ \vdots & & & \ddots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

and define

$$a_{i\cdot} = \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \\ \vdots \\ a_{in} \end{pmatrix} \quad i = 1, 2, \ldots, m \quad \text{and} \quad a_{\cdot j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix} \quad j = 1, 2, \ldots, n\ .$$

(a) Define $F_i : \mathbb{R}^n \to \mathbb{R}$ by $F_i(x) := a_{i\cdot}^T x,\ i = 1, 2, \ldots, m$. What are $\nabla F_i(x)$ and $\nabla^2 F_i(x)$?

(b) Define $h_i : \mathbb{R}^n \to \mathbb{R}$ by $h_i(x) := (a_{i\cdot}^T x - b_i)^2/2,\ i = 1, 2, \ldots, m$. What are $\nabla h_i(x)$ and $\nabla^2 h_i(x)$?
(c) Define $F : \mathbb{R}^n \to \mathbb{R}^m$ by $F(x) := [F_1(x), \ldots, F_m(x)]^T$. What is the Jacobian matrix for F?
(d) Define $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) = \sum_{i=1}^m h_i(x)$. Show that $h(x) = \frac{1}{2}\|F(x)\|_2^2 = \frac{1}{2}\|Ax - b\|_2^2$.
(e) Show that $A^T A = \sum_{i=1}^m a_{i\cdot} a_{i\cdot}^T$.
(f) Given h as defined in (d) above, what are $\nabla h(x)$ and $\nabla^2 h(x)$?