Math 408 Homework Set 1

Linear Algebra Review Problems

(1) Consider the system

$$\begin{array}{rcrcrcr} 4x_1 & & & - & x_3 & = & 200 \\ 9x_1 & + & x_2 & - & x_3 & = & 200 \\ 7x_1 & - & x_2 & + & 2x_3 & = & 200\ . \end{array}$$

Solution: $(x_1, x_2, x_3)^T = (30,\ -150,\ -80)^T$

(2) Represent the linear span of the four vectors

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 1 \\ 7 \\ 1 \end{bmatrix}, \quad \text{and} \quad x_4 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 5 \end{bmatrix},$$

as the range space of some matrix.

Solution: $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 2 & 1 & 7 & 0 \\ 1 & -2 & 1 & 5 \end{bmatrix}$

(3) Compute a basis for $\mathrm{nul}\,(A^T)^\perp$ where A is given by

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 2 & 1 & 7 & 0 \\ 1 & -2 & 1 & 5 \end{bmatrix}.$$

Solution: Since $\mathrm{nul}\,(A^T)^\perp = \mathrm{Ran}\,(A)$, we only need to row reduce A^T to get the basis $u_1 := (1, 0, 2, 0)^T,\ u_2 := (0, 1, 3, 0)^T,\ u_3 := (0, 0, 0, 1)$.

(4) Find the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -4 & 1 \\ 0 & 2 & 1 \end{pmatrix}$. **Solution:**$B^{-1} = \frac{1}{4}\begin{pmatrix} 6 & 2 & -2 \\ -1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$.

(5) Solve the following system of linear equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & & & = & 1 \\ -x_1 & - & 4x_2 & + & x_3 & = & 2 \\ & & 2x_2 & + & x_3 & = & 0. \end{array}$$

Solution: $B^{-1}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 10 \\ -3 \\ 6 \end{pmatrix}$.

(6) Determine whether the following system of linear equations has a solution or not.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 0 \end{pmatrix}.$$

Solution: No solution exists since the augmented matrix can be reduced to

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(7) Find a 2 by 2 square matrix B satisfying

$$A = B \cdot C,$$

where $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & -3 & 0 \\ 8 & 9 & 3 \end{pmatrix}$.

Solution: $B = \frac{1}{3}\begin{bmatrix} -3 & 0 \\ 2 & 1 \end{bmatrix}$.

(8) Show that the Gaussian elimination matrix for the vector

$$v = \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

where the pivot $\alpha \in \mathbb{R}$ is non-zero, $a \in \mathbb{R}^k$, and $b \in \mathbb{R}^{(n-(k+1))}$ is non-singular by providing an expression for its inverse.

(9) What is the Gaussian elimination matrix for the vector

$$v = \begin{bmatrix} 2 \\ -10 \\ 16 \\ 2 \end{bmatrix}?$$

where the entry $x_1 = 2$ is the pivot? What is it if the pivot is $x_2 = -10$?

Solution: $x_2 = -10$

$$\text{Gauss: } G := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 8/5 & 1 & 0 \\ 0 & 1/5 & 0 & 1 \end{bmatrix} \qquad \text{Gauss-Jordan: } J := \begin{bmatrix} 1 & 1/5 & 0 & 0 \\ 0 & -1/10 & 0 & 0 \\ 0 & 8/5 & 1 & 0 \\ 0 & 1/5 & 0 & 1 \end{bmatrix}$$

(10) Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{k\times m}$ where a_{ij} and b_{ij} is the ij^{th} elements of A and B, respectively. Denote the i^{th} row of A by $a_{i\cdot}$ (a row vector) and the j^{th} column of B by $b_{\cdot j}$ (a column vector). By construction, $A^TB \in \mathbb{R}^{n\times m}$. If $(A^TB)_{ij}$ is the ij^{th} element if A^TB, show that $(A^TB)_{ij} = (a_{i\cdot}b_{\cdot j})$.

Solution: By the definition of matrix multiplication, $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i\cdot}b_{\cdot j}$ which yields the result.

(11) Show that the product of two lower triangular $n \times n$ matrices is always a lower triangular matrix.

Solution: Let $A, B \in \mathbb{R}^{n\times n}$ be lower triangular. Let $a_{i\cdot}$ be the rows of A, $i = 1, \ldots, n$, and let $b_{\cdot j}$ be the columns of B, $j = 1, \ldots, n$. Since A and B are lower triangular, we have $a_{ik} = 0$ for $k > i$ and $b_{kj} = 0$ for $k < j$, for all $i, j \in \{1, 2, \ldots, n\}$. Let $i < j$. By the previous problem,

$$(AB)_{ij} = a_{i\cdot}b_{\cdot j} = \sum_{k=1}^{j-1} a_{ik}b_{kj} + \sum_{k=j}^{n} a_{ik}b_{kj} \ .$$

Since $k < j$ in the first summand above, $b_{kj} = 0$ for $k = 1, \ldots, j-1$ so that the first summand is zero. Also, since $k \geq j > i$ in the second summand above, $a_{ik} = 0$ for $k = j, \ldots, n$ so that the second summand is zero as well. Consequently, $(AB)_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i < j$, or equivalently, AB is lower triangular.

(12) Show that the inverse of a non-singular lower triangular matrix is always lower triangular.

Solution: Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonsingular lower triangular matrix so that $0 \neq \det A = a_{11}a_{22}\cdots a_{nn}$. Consequently, the diagonal of A has no zeros. Let B be the inverse of A. If B is not lower triangular, then there is a smallest $i_0 \in \{1, \ldots, n-1\}$ for which there is a $j_0 > i_0$ with $b_{i_0 j_0} \neq 0$. By definition, $b_{sj_0} = 0$ for $s < i_0$. Then, as above,

$$0 = (AB)_{i_0 j_0} = a_{i_0\cdot}b_{\cdot j_0} = \sum_{s=1}^{i_0-1} a_{j_0 s}b_{si_0} + a_{i_0 i_0}b_{i_0 j_0} + \sum_{s=i_0+1}^{n} a_{i_0 s}b_{sj_0}.$$

Since $b_{sj_0} = 0$ for $s < i_0$, the first summand is zero. The second summand is also zero since $a_{i_0 s} = 0$ for $s > i_0$. But $a_{i_0 i_0} \neq 0$ and $b_{i_0 j_0} \neq 0$ so that $a_{i_0 i_0}b_{i_0 j_0} \neq 0$. This contradiction implies that no such $b_{i_0 j_0} \neq 0$ can exist, that is, B is lower triangular.

(13) A *Housholder transformation* on $\mathbb{R}^n$ is any $n \times n$ matrix of the form

$$P = I - 2\frac{vv^T}{v^Tv}$$

for some non-zero vector $v \in \mathbb{R}^n$. The Householder transformation is the reflection across the hyperplane $v^Tx = 0$.

(a) Given any two vectors u and w in $\mathbb{R}^n$ such that $\|u\| = \|w\|$ and $u \neq w$, if P is the Householder transformation based on the vector $v := u - w$, show that $Pu = w$.

Solution: Since $u \neq w$ and $\|u\| = \|w\|$, we have $0 \neq v^Tv = \|u\|^2 - 2w^Tu + \|w\|^2 = 2(\|u\|^2 - w^Tu)$. Hence $(I - 2\frac{vv^T}{v^Tv})u = u - 2(u-w)\frac{\|u\|^2 - w^Tu}{2(\|u\|^2 - w^Tu)} = w$.

(b) If

$$u = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

explicitly construct the Householder transformation for which $Pu = w$.

Solution: $P = -\begin{bmatrix} 1/2 & 1/6 & 5/6 & 1/6 \\ 1/6 & -53/54 & 5/54 & 53/54 \\ 5/6 & 5/54 & -29/54 & /54 \\ 1/6 & 1/54 & 5/54 & -53/54 \end{bmatrix}$.

(c) Show that every Householder transformation P satisfies $P^T = P$ and $P^2 = I$.

Multi-variable Calculus Review Problems

(1) Find the local and global minimizers and maximizers of the following functions.

(a) $f(x) = x^2 + 2x$

Solution: $f'(x) = 2(x+1),\ f'(x) = 2$. Global max at $x = -1$.

(b) $f(x) = x^2 e^{-x^2}$

Solution: $f(x) \geq 0\ \forall x$ and $f'(x) = 2xe^{-x^2}(1 - x^2)$. Global min at $x = 0$ and global max at $x = \pm 1$.

(c) $f(x) = x^2 + \cos x$

Solution: $f'(x) = 2x - \sin x,\ f'(x) = 2 - \cos x > 0\ \forall x$. Global min at $x = 0$.

(d) $f(x) = x^3 - x$

Solution: $f(x) = x(x-1)(x_1),\ f'(x) = 3x^2 - 1,\ f'(x) = 6x$. Local max at $x = -\frac{1}{\sqrt{3}}$, local min at $x = \frac{1}{\sqrt{3}}$.

(2) and (3) Compute the gradient and Hessian of the following functions. In the solutions below, for $x \in \mathbb{R}^n$, $\mathrm{diag}\,(x)$ is the $n \times n$ diagonal matrix with diagonal entries given by the vector x in the order given.

(a) $f(x) = x_1^3 + x_2^3 - 3x_1 - 15x_2 + 25 : \quad f : \mathbb{R}^2 \to \mathbb{R}$

Solution:$\nabla f(x) = \begin{pmatrix} 3x_1^2 - 3 \\ 3x_2 - 15 \end{pmatrix},\ \nabla^2 f(x) = 6\mathrm{diag}\,(x_1,\ x_2)$

(b) $f(x) = x_1^2 + x_2^2 - \sin(x_1 x_2) \qquad f : \mathbb{R}^2 \to \mathbb{R}$

Solution:$\nabla f(x) = \begin{pmatrix} 2x_1 - x_2 \cos(x_1 x_2) \\ 2x_2 - x_1 \cos(x_1 x_2) \end{pmatrix},\ \nabla f(x) = \big(2 + x_2^2 \sin(x_1 x_2) \quad x_1 x_2 \sin(x_1 x_2) - \cos(x_1 x_2$

(c) $f(x) = \|x\|^2 = \sum_{j=1}^n x_j^2 : \qquad f : \mathbb{R}^n \to \mathbb{R}$

Solution:$\nabla f(x) = 2x,\ \nabla^2 f(x) = 2I$

(d) $f(x) = e^{\|x\|^2}$

Solution:$\nabla f(x) = 2xe^{\|x\|^2},\ \nabla^2 f(x) = 2e^{\|x\|^2}(I + 2xx^T)$

(e) $f(x) = x_1 x_2 x_3 \cdots x_n : \qquad f : \mathbb{R}^n \to \mathbb{R}$

Solution: $\nabla f(x) = \begin{pmatrix} x_2 x_3 \cdots x_n \\ x_1 x_3 \cdots x_n \\ \vdots \\ x_1 x_2 \cdots x_{n-1} \end{pmatrix},\ \nabla^2 f(x) = \begin{pmatrix} 0 & x_3 x_2 \cdots x_n & x_2 x_4 \cdots x_n & \cdots & x_2 x_3 \cdots x_{n-1} \\ x_3 x_4 \cdots x_n & 0 & x_1 x_3 \cdots x_n & \cdots & x_1 x_3 \cdots x_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_2 \cdots x_{n-1} & x_1 x_3 \cdots x_{n-1} & \cdots & \cdots & 0 \end{pmatrix}$

(f) $f(x) = -\log(x_1 x_2 x_3 \cdots x_n)$ for $x_j > 0,\ j = 1, \ldots n$, and undefined otherwise: $\qquad f : \mathbb{R}^n \to \mathbb{R}$. Compute $\nabla f(x)$ for $x_j > 0,\ j = 1, \ldots n$.

Solution:$\nabla f(x) = (-1/x_1,\ -1/x_2,\ \cdots,\ -1/x_n)^T,\ \nabla^2 f(x) = \mathrm{diag}\,(1/x_1^2,\ 1/x_2^2, \cdots,\ 1/x_n^2)$

(4) Let $b \in \mathbb{R}^m$ and consider the matrix $A \in \mathbb{R}^{m \times n}$ given by

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & a_{2n} \\ \vdots & & & \ddots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & \dots & a_{mn} \end{bmatrix}$$

and define the column vectors

$$a_{i\cdot} := \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \\ \vdots \\ a_{in} \end{pmatrix} \quad i = 1, 2, \dots, m \quad \text{and} \quad a_{\cdot j} := \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix} \quad j = 1, 2, \dots, n \ .$$

(a) Define $F_i : \mathbb{R}^n \to \mathbb{R}$ by $F_i(x) := a_{i\cdot}^T x,\ i = 1, 2, \dots, m$. What are $\nabla F_i(x)$ and $\nabla^2 F_i(x)$?

Solution: $\nabla F_i(x) = a_{i\cdot},\ \nabla^2 F_i(x) = 0$

(b) Define $h_i : \mathbb{R}^n \to \mathbb{R}$ by $h_i(x) := (a_{i\cdot}^T x - b_i)^2/2,\ i = 1, 2, \dots, m$. What are $\nabla h_i(x)$ and $\nabla^2 h_i(x)$?

Solution: $\nabla h_i(x) = (a_{i\cdot}^T x - b_i) a_{i\cdot} = a_{i\cdot} a_{i\cdot}^T x - a_{i\cdot} b_i,\ \nabla^2 h_i(x) = a_{i\cdot} a_{i\cdot}^T$

(c) Define $F : \mathbb{R}^n \to \mathbb{R}^m$ by $F(x) := [F_1(x), \dots, F_m(x)]^T$. What is the Jacobian matrix for F?

Solution: $\nabla F(x) = A$

(d) Define $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) = \sum_{i=1}^m h_i(x)$. Show that $h(x) = \frac{1}{2}\|F(x)\|_2^2 = \frac{1}{2}\|Ax - b\|_2^2$.

Solution: $h(x) = \sum_{i=1}^m h_i(x) = \frac{1}{2}\sum_{i=1}^m (a_{i\cdot}^T x - b_i)^2 = \frac{1}{2}\|Ax - b\|^2$.

(e) Show that $A^T A = \sum_{i=1}^m a_{i\cdot} a_{i\cdot}^T$.

Solution: $ATA = \begin{bmatrix} a_{1\cdot} & a_{2\cdot} & \cdots & a_{m\cdot} \end{bmatrix} \begin{bmatrix} a_{1\cdot}^T \\ a_{2\cdot}^T \\ \vdots \\ a_{m\cdot}^T \end{bmatrix} = \sum_{i=1}^m a_{i\cdot} a_{i\cdot}^T$

(f) Given h as defined in (d) above, what are $\nabla h(x)$ and $\nabla^2 h(x)$?

Solution: $\nabla h(x) = \sum_{i=1}^m \nabla h_i(x) = \sum_{i=1}^m a_{i\cdot} a_{i\cdot}^T x - a_{i\cdot} b_i = (\sum_{i=1}^m a_{i\cdot} a_{i\cdot}^T) x - A^T b = A^T A x - A^T b = A^T(Ax - b)$ and $\nabla^2 h(x) = A^T A$.