Math 407 Linear Least Squares Homework

This homework set will focus on the linear least squares problem

$$\mathcal{LLS} \qquad \min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax-b\|^2 \ ,$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

(1) Listed below are two functions. In each case write the problem $\min_x f(x)$ as a linear least squares problem by specifying the matrix A and the vector b, and then solve the associated problem.
  (a) $f(x) = (2x_1 - x_2 + 1)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$
  (b) $f(x) = (1 - x_1)^2 + \sum_{j=1}^{5-1}(x_j - x_{j+1})^2$

(2) Consider the data points $(x, y) \in \mathbb{R}$, $(1, 1)$, $(2, 0)$, $(-1, 2)$, and $(0, -1)$. We wish to determine a real polynomial of degree 2 that best fits this data. A general real polynomial of degree 2 has the form $p(\lambda) = x_0 + x_1\lambda + x_2\lambda^2$, where $x = (x_0, x_1, x_2)^T \in \mathbb{R}^3$. Note that there are more data points that there are unknown coefficients x_0, x_1, and x_2 and so it is unlikely that there exists a second degree polynomial that fits this data precisely.
  (a) Write the problem of determining the quadratic polynomial that "best" fits this data as a linear least squares problem by specifying the matrix A and the vector b.
  (b) Solve this linear least squares problem.

(3) Find the quadratic polynomial $p(t) = x_0 + x_1 t + x_2 t^2$ that best fits the following data in the least-squares sense:

t	-2	-1	0	1	2
y	2	-10	0	2	1

(4) Consider the problem $\mathcal{LLS}$ with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

  (a) What are the normal equations for this A and b.
  (b) Solve the normal equations to obtain a solution to the problem $\mathcal{LLS}$ for this A and b.
  (c) What is the general reduced QR factorization for this matrix A?
  (d) Compute the orthogonal projection onto the range of A.
  (e) Use the recipe

$$\begin{aligned} AP &= Q[R_1 \ R_2] && \text{the general reduced QR factorization} \\ \hat{b} &= Q^T b && \text{a matrix-vector product} \\ \bar{w}_1 &= R_1^{-1}\hat{b} && \text{a back solve} \\ \bar{x} &= P\begin{bmatrix} R_1^{-1}\hat{b} \\ 0 \end{bmatrix} && \text{a matrix-vector product.} \end{aligned}$$

  to solve $\mathcal{LLS}$ for this A and b.
  (f) If $\bar{x}$ solves $\mathcal{LLS}$ for this A and b, what is $A\bar{x} - b$?

(5) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

   (a) Compute the orthogonal projection onto $\mathrm{Ran}(A)$.
   (b) Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.

(6) Let $A \in \mathbb{R}^{m\times n}$. Show that $\mathrm{Null}(A) = \mathrm{Null}(A^T A)$.

(7) Let $A \in \mathbb{R}^{m\times n}$ be such that $\mathrm{Null}(A) = \{0\}$.
   (a) Show that $A^T A$ is invertible.
   (b) Show that the orthogonal projection onto $\mathrm{Ran}(A)$ is the matrix $P := A(A^T A)^{-1} A^T$.