Computing Dual LPs without Conversion to Standard Form

(1) Compute the dual LP to each of the following LPs without first converting to standard form.

(a)

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 10x_3 \\ \text{subject to} & x_1 + x_2 - x_3 = 12 \\ & x_1 - x_2 + x_3 \le 8 \\ & 0 \le x_2 \le 10 \end{array}$$

(b)

$$\begin{array}{llllll} \text{maximize} & & 42x_2 & -30x_3 & & \\ \text{subject to} & x_1 & -x_2 & +x_3 & -x_4 & = 0 \\ & x_1 & & +x_3 & -x_4 & \le 5 \\ & & 5x_2 & +x_3 & -5x_4 & = -1 \\ & 0 & \le x_1, & & 0 \le & x_3 \le 20 \end{array}$$

(2) Consider the mini–max problem

$$\min_{x \in I\!R^n} \max_{i=1,2,\ldots,m} \{a_i^T x - b_i\}$$

where $a_i \in I\!R^n$ and $b_i \in I\!R$ for $i = 1, 2, \ldots, m$.

(a) Show that this mini-max problem is in some sense *equivalent* to the LP

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & Ax - b \le x_0 e, \end{array} \tag{1}$$

where $A = (a_{ij})_{m\times n}$, $b = [b_1, b_2, \ldots, b_m]^T$, and $e \in I\!R^m$ is the vector of all ones.

(b) Show that the dual of the LP (1) is

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y = 0,\ e^T y = 1, \\ & 0 \le y \end{array}$$

(3) Consider the system of linear inequalities and equations

$$Ax \le b, \qquad Bx = d, \tag{2}$$

where $A \in I\!R^{m\times n}$, $B \in I\!R^{s\times t}$, $d \in I\!R^s$, and $b \in I\!R^n$. We are interested in studying the consistency of this system, that is, we are interested in determining conditions under which the solution set $S = \{x : Ax \le b,\ Bx = d\}$ is non–empty. For this purpose, we make use of the following linear program:

$$\begin{array}{rlrl} \mathcal{P}: & \text{maximize} & -e^T z & \\ & & Ax - z & \leq b \\ & & Bx & = d \\ & & 0 & \leq z \end{array}$$

where $e \in I\!R^m$ is the vector of all ones ($e = (1, 1, 1, \ldots, 1)^T$).

(a) Show that the system (2) is consistent (i.e. $S \neq \emptyset$) if and only if the optimal value in $\mathcal{P}$ is zero.

(b) Show that the dual to the LP $\mathcal{P}$ is the LP

$$\begin{array}{rll} \mathcal{D}: & \text{minimize} & b^T u + d^T v \\ & & A^T u + B^T v = 0 \\ & & 0 \leq u \leq e. \end{array}$$

(c) Show that the system $Ax \leq b$ is inconsistent (i.e. $S = \emptyset$) if and only if there are vectors $u \in I\!R^m$ and $v \in I\!R^s$ such that $0 \leq u,\ A^T u + B^T v = 0$, and $b^T u + d^T v < 0$.

Solution to 2.b: The primal problem can be written as

$$\begin{array}{ll} \max & \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T \begin{pmatrix} x_0 \\ x \end{pmatrix} \\ \text{s.t.} & [-e\ A] \begin{pmatrix} x_0 \\ x \end{pmatrix} \leq b\ . \end{array}$$

Therefore the dual objective is $b^T y$. The primal variables are free, so the dual contains only the linear equality $\begin{bmatrix} -e^T \\ A^T \end{bmatrix} y = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The primal only has linear inequalities so the dual variables are non-negative: $0 \leq y$. Consequently, the dual is

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & \begin{bmatrix} -e^T \\ A^T \end{bmatrix} y = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ & 0 \leq y\ , \end{array}$$

which is equivalent to the given dual.