MIDTERM EXAM GUIDE

Calculators are not allowed for this exam. The exam will consist of 5 questions. The midterm is worth a total of 350 points. The content of each question and its point value is listed below.

Question 1: (70 points) In this question you will be asked to recite and/or use the definition of one or more terms from the class notes, Sections 1–4 and Section 4 up to *primal simplex pivot* (but not the following terms starting with *dual simplex pivot*). The list of terms that you need to know can be found in the weekly overviews for weeks 1–5.

Question 2: (50 points) In this question you will be asked to solve a two dimensional LP graphically. For this you must include *all* of the elements discussed in class and in the notes. Please bring a straight-edge and whatever other graphing aids you require.

Question 3: (70 points) In this question you will be asked to model a given problem as an LP. The particular model will be chosen from the models 1 through 12 found on the class webpage.

Question 4: (80 points) In this question you will be asked to solve one or more LPs using the two phase simplex algorithm in tableau format.

Question 5: (80 points) In this question you will be asked to do one or more (perhaps all four) of the following: (a) convert an LP to standard form, (b) apply the Complementary Slackness Theorem to determine if a given point is the solution to a given LP (c) give the dual of an LP that is in generalized standard form, and (d) state and/or prove one or more of the following theorems for LPs in standard form (*not* generalized standard form): (i) the Weak Duality Theorem, (ii) the Fundamental Theorem of Linear Programing, and (iii) the Strong Duality Theorem.

SAMPLE QUESTIONS

(1) The following questions refer to an LP in standard form¿

$$\mathcal{P} \qquad \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \le b,\ 0 \le x\ . \end{array}$$

- (a) What does it mean to say that $\mathcal{P}$ is unbounded? Provide and example of an unbounded LP.
- (b) What is the initial dictionary for this LP?
- (c) What is the initial tableau for this LP?
- (d) Under what conditions is the following system a dictionary for $\mathcal{P}$?

$$(D) \qquad \begin{array}{l} x_i = \hat{b}_i - \sum_{j\in N} \hat{a}_{ij} x_j \quad i \in B \\ z = \hat{z} + \sum_{j\in N} \hat{c}_j x_j . \end{array}$$

- (e) When is (D) primal feasible?
- (f) When is (D) dual feasible?
- (g) When is (D) primal degenerate?
- (h) When is (D) optimal?
- (i) When does (D) indicate that $\mathcal{P}$ is unbounded?
- (j) What is the relationship between dictionaries and simplex tableaus for LPs in standard form?
- (k) What is the dual for $\mathcal{P}$?
- (l) Give the block matrix equation illustrates the transition from the initial tableau to any intermediary tableau. Denote the intermediary tableau as T_k, where k stands for the kth tableau. (Hint: $GT_0 = T_k$)
- (m) Under what conditions is this initial tableau T_0 primal feasible?
- (n) Under what conditions is this intermediary tableau T_k primal feasible?
- (o) Under what conditions is this intermediary tableau T_k dual feasible?
- (p) Under what conditions is this intermediary tableau T_k primal degenerate?

(q) Under what conditions is this intermediary tableau T_k optimal?

(r) State the auxiliary problem for an LP in standard form and explain what it is used for. In addition, show that the auxiliary problem is always feasible and bounded.

(2) Consider the following LP:

$$\begin{array}{lrcrcl}
\text{maximize} & 4x & - & y & & \\
\text{subject to} & -2x & + & y & \leq & 4 \\
 & x & + & y & \leq & 7 \\
 & 2x & - & y & \leq & 1 \\
 & x & + & y & \geq & 1 \\
 & x & \leq & 2, & 0 & \leq \; y
\end{array}$$

Solve this LP graphically using the technique described in the class notes. For full credit you will need to accurately graph all constraints (along with little arrows indicating the correct side), the feasible region, the objective normal, the give the solution (with numerical coordinates), and the optimal value.

(3) A cargo plane has three compartments for storing cargo: front, center, and back. These compartments have capacity limits on both *weight* and *space*, as summarized below:

Compartment	*Weight capacity (tons)*	*Space capacity (cu ft)*
Front	12	7,000
Center	18	9,000
Back	10	5,000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane.
The following four cargoes have been offered for shipment on an upcoming flight as space is available:

Cargo	*Weight (tons)*	*Volume (cu ft/ton)*	*Profit ($/ton)*
1	20	500	280
2	16	700	360
3	25	600	320
4	13	400	250

Any portion of these cargoes can be accepted. The object is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight. Model this problem as an LP.

(4) Solve the following LPs using the simplex algorithm in *tableau format* (zero credit will be given for solutions using dictionary format or by graphing). State the solution, the solution to the dual, as well as the associated optimal value.

(a)

$$\begin{array}{lrcrcrcl}
\text{maximize} & & & 2x_2 & + & 3x_3 & & \\
\text{subject to} & -x_1 & + & x_2 & - & x_3 & \leq & 1 \\
 & x_1 & - & 2x_2 & & & \leq & 0 \\
 & x_1 & & & + & 3x_3 & \leq & 1 \\
 & 0 & \leq & x_1, & x_2, & x_3 & &
\end{array}$$

(b)

$$\begin{array}{lrcrcr} \text{maximize} & 3x_1 & - & x_2 & & \\ \text{subject to} & x_1 & - & 4x_2 & \leq & -2 \\ & -2x & - & x_2 & \leq & -5 \\ & 2x & + & x_2 & \leq & 14 \\ & 0 & \leq & x_1, & x_2 & \end{array}$$

(c)

$$\begin{array}{lrcrcrcrcr} \text{maximize} & 4x_1 & + & 4x_2 & + & 5x_3 & + & 3x_4 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 40 \\ & x_1 & + & x_2 & + & 2x_3 & + & x_4 & \leq & 40 \\ & 2x_2 & + & 2x_2 & + & 3x_3 & + & x_4 & \leq & 60 \\ & 3x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & \leq & 50 \\ & 0 & \leq & x_1, & x_2, & x_3, & x_4. & & & \end{array}$$

(d)

$$\begin{array}{lrcrcrl} \text{maximize} & -x_1 & - & 2x_2 & - & 3x_3 & \\ & x_1 & + & 2x_2 & - & 2x_3 & \leq -2 \\ & & - & x_2 & + & x_3 & \leq 4 \\ & -x_1 & - & x_2 & + & x_3 & \leq -1 \\ & 0 & \leq & x_1, & x_2, & x_3 & \end{array}$$

(5) (a) Put the following LP into standard form:

$$\begin{array}{lrcrcrcrcr} \text{minimize} & 4x_1 & - & 2x_2 & + & x_3 & & & & \\ & -x_1 & + & 3x_2 & - & x_3 & \geq & -1 & & \\ & & & 5x_2 & + & 3x_3 & = & 5 & & \\ & x_1 & + & x_2 & + & x_3 & \leq & 1 & & \\ & -1 & \leq & x_2, & -2 & \leq & x_3 & \leq & 2 & \end{array}$$

(b) Use the Complementary Slackness Theorem to determine if the vector $x = (0, 5, 0, 1, 1)^T$ solves the LP

$$\begin{array}{lrcrcrcrcrcr} \text{maximize} & & & x_2 & & & + & 5x_4 & + & 5x_5 & & \\ \text{subject to} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & & & \leq & 11 \\ & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & + & x_5 & \leq & 10 \\ & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & \leq & -2 \\ & x_1 & & & & & + & x_4 & + & 3x_5 & \leq & 4 \\ & 0 & \leq & x_1, & x_2, & x_3, & x_4, & x_5\ . & & & & \end{array}$$

(c) Formulate a dual for the following LPs.

(i)

$$\begin{array}{lrcl} \text{minimize} & c^T x & & \\ \text{subject to} & Ax & \leq & 0 \\ & Bx & = & 0, \end{array}$$

where $c \in I\!R^n,\ A \in I\!R^{s\times n}$, and $B \in I\!R^{t\times n}$.

(ii)

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 10x_3 \\ \text{subject to} & x_1 + x_2 - x_3 = 12 \\ & x_1 - x_2 + x_3 \leq 8 \\ & 0 \leq x_2 \leq 10 \end{array}$$

(d) For an LP in standard form answer the following.
  (i) State and prove the Weak Duality Theorem.
  (ii) State and prove the Fundamental Theorem of Linear Programming.
  (iii) State and prove the Strong Duality Theorem.