1. (a) What does it mean to say that an LP is unbounded? Provide and example of an unbounded LP.

   Solution: It is feasible and the optimal value is infinite ($+\infty$ for maximization and $-\infty$ for minimization).
   Example: $\max x$ s.t. $0 \le x$.

   (b) What is a dictionary for an LP in standard form?

   Solution: A dictionary for the LP $\max \left\{c^T x \mid Ax \le b,\ 0 \le x\right\}$, where $A \in I\!R^{m\times n},\ c \in I\!R^n$, and $b \in I\!R^m$, is a linear system of the form

$$D_B \left\{ \begin{array}{rcll} x_i = & \hat{b}_i - & \sum_{j\in N}^{n} \hat{a}_{ij} x_j & \qquad i \in B \\ z = & \hat{z} + & \sum_{j\in N}^{n} \hat{c}_j x_j & \end{array} \right.$$

   having the same set of solutions as the system

$$D_{\text{initial}} \left\{ \begin{array}{rcll} x_{n+i} = & b_i - & \sum_{j=1}^{n} a_{ij} x_j & \qquad i = 1,\dots,m \\ & & & \\ z = & & \sum_{j=1}^{n} c_j x_j & \end{array} \right.$$

   where the index sets B and N satisfy $B \cap N = \emptyset,\ B \cup N = \{1,\dots,n+m\}$ with $|B| = m$ and $|N| = n$.

   (c) $\begin{bmatrix} 0 & A & I & b \\ -1 & c^T & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix}$.

   (d) The index sets B and N satisfy $B \cap N = \emptyset,\ B \cup N = \{1,\dots,n+m\}$ with $|B| = m$ and $|N| = n$, and the set of solutions to (D) coincide with those of the initial dictionary for $\mathcal{P}$.

   (e) $\hat{b}_i \ge 0\ \forall\, i \in B$.

   (f) $\hat{c}_i \le 0\ \forall\, j \in N$.

   (g) There exists $i \in B$ such that $\hat{b}_i = 0$.

   (h) It is both primal and dual feasible, i.e., $\hat{b}_i \ge 0\ \forall\, i \in B$ and $\hat{c}_i \le 0\ \forall\, j \in N$, respectively.

   (i) There is an $j_0 \in N$ such that $\hat{c}_{j_0} > 0$ and $\hat{a}_{ij_0} \le 0\ \forall\, i \in B$.

   (j) The tableau is the augmented matrix for its associated dictionary.

   (k)

$$\mathcal{D} \quad \begin{array}{ll} \text{minimize} & b^T x \\ \text{subject to} & A^T y \ge c,\ 0 \le y\ . \end{array}$$

   (l) Give the block matrix equation that illustrates the transition from the initial tableau to any intermediary tableau.

Solution:

$$GT_0 = T_k$$

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} 0 & A & I & b \\ -1 & c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & RA & R & Rb \\ -1 & c^T - y^T A & -y^T & -y^T b \end{bmatrix} \quad \text{or}$$

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} RA & R & Rb \\ c^T - y^T A & -y^T & -y^T b \end{bmatrix}$$

(m) Under what conditions is the initial tableau T_0 primal feasible?

Solution: $b \geq 0$

(n) Under what conditions is the intermediary tableau T_k primal feasible?

Solution: $Rb \geq 0$

(o) Under what conditions is the intermediary tableau T_k dual feasible?

Solution: $\begin{bmatrix} c^T - y^T A, & -y^T \end{bmatrix} \leq 0$

(p) Under what conditions is the intermediary tableau T_k primal degenerate?

Solution: $Rb \geq 0$ and $\exists\, i \in \{1, 2, \dots, m\}$ s.t. $(Rb)_i = 0$

(q) Under what conditions is the intermediary tableau T_k dual degenerate?

Solution: $(c^T - y^T A,\ -y^T) \leq 0$ and $\exists$ a nonbasic $i_0 \in \{1, 2, \dots, n+m\}$ s.t. $(c^T - y^T A,\ -y^T)_{i_0} = 0$

(r) Under what conditions is the intermediary tableau T_k optimal?

Solution: Both primal and dual feasible,i.e. $Rb \geq 0$ and $\begin{bmatrix} c^T - y^T A & -y^T \end{bmatrix} \leq 0$

(s) State the auxiliary problem for an LP in standard form and explain what it is used for. In addition, show that the auxiliary problem is always feasible and bounded.

Solution: The auxiliary problem is $\max\{-x_0 \mid Ax - x_0\mathbf{1} \leq b,\ 0 \leq x, x_0\}$ (or equivalently, $\min\{x_0 \mid Ax - x_0\mathbf{1} \leq b,\ 0 \leq x, x_0\}$), where $\mathbf{1}$ is the vector of all ones. It is always feasible since $(x, x_0) = (0, \max\{0, -b_1, -b_2, \dots, -b_m\})$ is always a feasible point. It is always bounded since zero is an upper bound.

2. Solution not provided.

3. A cargo plane has three compartments for storing cargo: front, center, and back. These compartments have capacity limits on both *weight* and *space*, as summarized below:

Compartment	*Weight capacity (tons)*	*Space capacity (cu ft)*
Front	12	7,000
Center	18	9,000
Back	10	5,000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane.

The following four cargoes have been offered for shipment on an upcoming flight as space is available:

Cargo	*Weight* *(tons)*	*Volume* *(cu ft/ton)*	*Profit* *($/ton)*
1	20	500	280
2	16	700	360
3	25	600	320
4	13	400	250

Any portion of these cargoes can be accepted. The object is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight. Model this problem as an LP.

Solution: Decision variables:C_{ij} =tons of cargo i in compartment j, $i = 1, 2, 3, 4$, $j = 1, 2, 3$ with $j = 1$ the front, $j = 2$ the middle, and $j = 3$ the back.

Objective: maximize profit $= 280\sum_{j=1}^{3} C_{1j} + 360\sum_{j=1}^{3} C_{2j} + 320\sum_{j=1}^{3} C_{3j} + 250\sum_{j=1}^{3} C_{4j}$

Constraints:
cargo availability: $\sum_{j=1}^{3} C_{1j} \le 20,\ \sum_{j=1}^{3} C_{2j} \le 16\ \ \sum_{j=1}^{3} C_{3j} \le 25,\ \sum_{j=1}^{3} C_{4j} \le 13$
Compartment weight and space capacities: $\sum_{i=1}^{4} C_{i1} \le 12,\ \sum_{i=1}^{4} C_{i2} \le 18,\ \sum_{i=1}^{4} C_{i3} \le 10$

$$\begin{pmatrix}500\\700\\600\\400\end{pmatrix}^T \begin{pmatrix}C_{11}\\C_{21}\\C_{31}\\C_{41}\end{pmatrix} \le 7,000, \quad \begin{pmatrix}500\\700\\600\\400\end{pmatrix}^T \begin{pmatrix}C_{12}\\C_{22}\\C_{32}\\C_{42}\end{pmatrix} \le 9,000, \quad \begin{pmatrix}500\\700\\600\\400\end{pmatrix}^T \begin{pmatrix}C_{13}\\C_{23}\\C_{33}\\C_{43}\end{pmatrix} \le 5,000,$$

Balance:

$$\frac{\sum_{i=1}^{4} C_{i1}}{12} = \frac{\sum_{i=1}^{4} C_{i2}}{18} = \frac{\sum_{i=1}^{4} C_{i3}}{10}$$

implicit: all variables non-negative

4. Solve the following LPs using the simplex algorithm in *tableau format* (zero credit will be given for solutions using dictionary format or by graphing). State the solution, the solution to the dual, as well as the associated optimal value.

   (a)

$$\begin{array}{lrcrcrcl}
\text{maximize} & & & 2x_2 & + & 3x_3 & & \\
\\
\text{subject to} & -x_1 & + & x_2 & - & x_3 & \le & 1 \\
 & x_1 & - & 2x_2 & & & \le & 0 \\
 & x_1 & & & + & 3x_3 & \le & 1 \\
 & 0 & \le & x_1, & x_2, & x_3 & &
\end{array}$$

   Solution: primal optimal solution $(1, 2, 0)^T$, dual optimal solution $(2, 0, 2)^T$, optimal value 4.

   (b)

$$\begin{array}{lrcrcr}
\text{maximize} & 3x_1 & - & x_2 & & \\
\\
\text{subject to} & x_1 & - & 4x_2 & \le & -2 \\
 & -2x & - & x_2 & \le & -5 \\
 & 2x & + & x_2 & \le & 14 \\
 & 0 & \le & x_1, & x_2 &
\end{array}$$

Solution: primal optimal solution $(6,2)^T$, dual optimal solution $(5/9,0,11/9)^T$, optimal value 16

(c)

$$\begin{array}{lrcrcrcrcl} \text{maximize} & 4x_1 & + & 4x_2 & + & 5x_3 & + & 3x_4 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & \le & 40 \\ & x_1 & + & x_2 & + & 2x_3 & + & x_4 & \le & 40 \\ & 2x_2 & + & 2x_2 & + & 3x_3 & + & x_4 & \le & 60 \\ & 3x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & \le & 50 \\ & 0 & \le & x_1, & & x_2, & & x_3, & & x_4. \end{array}$$

Solution: Sorry, this is a difficult computation: $x = (0,5,15,5)^T$, $y = (0,0,1,1)^T$, $z = 110$ is one solution. The optimal solution set is given by

$$\left\{ \begin{pmatrix} 0 \\ 15 \\ 10 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \;|\, 0 \le t \le 5 \right\} .$$

(d)

$$\begin{array}{lrcrcrl} \text{maximize} & -x_1 & - & 2x_2 & - & 3x_3 & \\ & x_1 & + & 2x_2 & - & 2x_3 & \le -2 \\ & & - & x_2 & + & x_3 & \le 4 \\ & -x_1 & - & x_2 & + & x_3 & \le -1 \\ & 0 & \le & x_1, & & x_2, & x_3 \end{array}$$

Solution: use dual simplex to get $x = (4,0,3)^T$, $y = (4,0,5)^T$, $z = -13$

5. (a) Put the following LP into standard form:

$$\begin{array}{lrcrcrcrcl} \text{minimize} & 4x_1 & - & 2x_2 & + & x_3 & & & & \\ & -x_1 & + & 3x_2 & - & x_3 & \ge & -1 & & \\ & & & 5x_2 & + & 3x_3 & = & 5 & & \\ & x_1 & + & x_2 & + & x_3 & \le & 1 & & \\ & -1 & \le & x_2, & -2 & \le & x_3 & \le & 2 & \end{array}$$

Solution:

$$\begin{array}{lrcrcrcrcr} \text{maximize} & -4z_1^+ & + & 4z_1^- & + & 2z_2 & - & z_3 & & \\ s.t. & z_1^+ & - & z_1^- & - & 3z_2 & + & z_3 & \le & 0 \\ & & & & & 5z_2 & + & 3z_3 & \le & 16 \\ & & & & - & 5z_2 & - & 3z_3 & \le & -16 \\ & z_1^+ & - & z_1^- & + & z_2 & + & z_3 & \le & 4 \\ & & & & & & & z_3 & \le & 4 \\ & 0 \le & z_1^+, & 0 \le & z_1^-, & 0 \le & z_2, & 0 \le & z_3 & \end{array}$$

(b) Use the Complementary Slackness Theorem to determine if the vector $x = (0,5,0,1,1)^T$ solves the LP

$$\begin{array}{lrcrcrcrcrcl} \text{maximize} & & & x_2 & & & + & 5x_4 & + & 5x_5 & & \\ \text{subject to} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & & & \le & 11 \\ & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & + & x_5 & \le & 10 \\ & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & \le & -2 \\ & x_1 & & & & & + & x_4 & + & 3x_5 & \le & 4 \\ & 0 & \le & x_1, & & x_2, & & x_3, & & x_4, & & x_5 \end{array}$$

Solution: $x = (0,5,0,1,1)^T$ is primal feasible, and in checking this we see the second inequality is strict. Hence, if x is optimal, then the corresponding dual solution must have $y_2 = 0$. Moreover, since x_2, x_4, x_5 are positive, the dual solution must have equality in the second, fourth and fifth dual equations. Putting this all together we find that the corresponding dual solution y must satisfy the system

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \\ 5 \\ 0 \end{bmatrix}.$$

Since the solution to this system is $y = (11/5, 0/17/5, -3/5)$, this x cannot solve the primal as this y is not feasible.

(c) (i) $\max \left\{0 \mid c + A^T u + B^T v = 0,\ 0 \le u\right\}$

i.

$$\begin{array}{lrcrcrcr} \text{maximize} & 12y_1 & + & 8y_2 & + & 10y_3 & & \\ \text{subject to} & y_1 & + & y_2 & & & = & 2 \\ & y_1 & - & y_2 & + & y_3 & \ge & -3 \\ & -y_1 & + & y_2 & & & = & 10 \\ & 0 & \le & y_2, & & 0 & \le & y_3 \end{array}$$

(d) Solution given in course notes.