Math 407A: Linear Optimization

Professor James Burke

Math Dept, University of Washington

Midterm Exam Comments

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Use (A, b, c) above to state the structure of an LP in standard form with c used in the objective. Also state the form of the dual.

(Primal) $\mathcal{P}$

(Dual) $\mathcal{D}$

Dictionaries and Simplex Tableaus

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$$\text{s.t. } Ax \leq b$$

$$0 \leq x$$

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Give the initial dictionary for this LP.

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$$x_{n+i} = \quad b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad i = 1, \ldots, m$$

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What is the relationship between the initial dictionary and the initial simplex tableau?

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What is the relationship between the initial dictionary and the initial simplex tableau?

The tableau is the augmented matrix for the dictionary.

Dictionaries and Simplex Tableaus

The initial dictionary.

$$\begin{aligned} x_{n+i} &= b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad i = 1, \ldots, m \\ z &= \sum_{j=1}^{n} c_j x_j \end{aligned}$$

- How is the basic solution for this initial dictionary identified?

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Set the variables $x_j = 0$ $j = 1, \ldots, n$ which then specifies that $x_{n+i} = b_i$ $i = 1, \ldots m$.

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$b_i \geq 0 \; i = 1, \ldots, m$

• Can one always start the primal simplex algorithm on this dictionary?
NO! Need $b \geq 0$. The primal simplex algorithm **requires** primal feasibility for implementation.

Dictionaries and Simplex Tableaus

The initial tableau.

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right]$$

- How is the basic solution for this initial tableau identified?

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• How does one start the primal simplex algorithm on this tableau?
If $b \not\geq 0$ start phase 1 of the primal simplex algorithm and solve the auxiliary problem; else, proceed as usual by locating incoming column.

Dictionaries and Simplex Tableaus

What is the structure of a general dictionary for the LP

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where B is a basis for the LP, that is B and N form a partition of $\{1, \ldots, n+m\}$ with B having m elements, and the set of solutions to this linear system coincides with that of the initial dictionary.

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One! Every basis **uniquely** identifies an associated dictionary.

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• If $B \subset \{1, \ldots, n+m\}$ has m elements, is B a basis?
Not necessarily since solution sets may not coincide.

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$$D_B: \quad \begin{array}{rclll} x_i & = & \hat{b}_i - & \sum_{j \in N} \hat{a}_{ij} x_j & i \in B \\ z & = & v + & \sum_{j \in N} \hat{c}_j x_j & \end{array}$$

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- When is D_B optimal?

$\hat{b}_i \geq 0 \ i \in B$ and $\hat{c}_j \leq 0 \ j \in N$, i.e. it is **both** primal and dual feasible.

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$\hat{b}_i \geq 0$ $i \in B$ and $\hat{c}_j \leq 0$ $j \in N$, i.e. it is **both** primal and dual feasible.

- When is D_B primal degenerate?

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$\hat{b}_i \geq 0$ $i \in B$ and $\hat{c}_j \leq 0$ $j \in N$, i.e. it is **both** primal and dual feasible.
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Dictionaries and Simplex Tableaus

$$D_B: \quad \begin{array}{rcll} x_i & = & \hat{b}_i - & \sum_{j\in N} \hat{a}_{ij}x_j \quad i \in B \\ z & = & v + & \sum_{j\in N} \hat{c}_j x_j \end{array}$$

- What is the basic solution identified by D_B?

Set $x_j = 0\ j \in N$ so that $x_i = \hat{b}_i\ i \in B$.

- When is this basic solution a basic feasible solution? $\hat{b}_i \geq 0\ i \in B$.
- When is D_B primal feasible? $\hat{b}_i \geq 0\ i \in B$.
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$\hat{b} \geq 0$ and $\exists\, j_0 \in N$ s.t. $\hat{c}_{j_0} > 0$ with $\hat{a}_{ij_0} \leq 0$ $\forall\, i \in B$.

Dictionaries and Simplex Tableaus

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Dictionaries and Simplex Tableaus

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$\hat{b} \geq 0$ and $\exists\, j_0 \in N$ s.t. $\hat{c}_{j_0} > 0$ with $\hat{a}_{ij_0} \leq 0 \ \forall\, i \in B$.

- What must be true about D_B to show that the LP is infeasible?

D_B can only show the LP is infeasible by showing that the dual is unbounded. That is, D_B is dual feasible ($\hat{c}_N \leq 0$) and $\exists\, i_0 \in B$ s.t. $\hat{b}_{i_0} < 0$ with $\hat{a}_{i_0 j} \geq 0 \ \forall\, j \in N$.

Dictionaries and Simplex Tableaus

What is the structure of a general simplex tableau for the LP

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \le b \\ & 0 \le x \end{aligned}$$

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$$\begin{aligned} &\max\ c^T x \\ &\text{s.t. } Ax \le b \\ &0 \le x \end{aligned}$$

The general simplex tableau T is obtained by multiplying the initial simplex tableau on the left by a product of Gauss-Jordan elimination matrices. It was shown in class that we can display this by the formula

$$T = \begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} 0 & A & I & b \\ -1 & c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & RA & R & Rb \\ -1 & c^T - y^T A & -y^T & -y^T b \end{bmatrix}$$

In particular, R is invertible. It is called the record matrix.

Dictionaries and Simplex Tableaus

Dictionary

$$\begin{array}{rcll} x_i & = & \hat{b}_i - & \sum_{j\in N} \hat{a}_{ij}x_j \quad i \in B \\ z & = & v + & \sum_{j\in N} \hat{c}_j x_j \end{array}$$

Associated tableau

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Dictionaries and Simplex Tableaus

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Dictionaries and Simplex Tableaus

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- Is $-y^T b > 0$ or is $-y^T b < 0$?

Dictionaries and Simplex Tableaus

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• If one starts the primal simplex algorithm from a primal feasible tableau, when is it possible for $(Rb)_i < 0$ for some i?
NEVER! The primal simplex algorithm only applies to primal feasible dictionaries and tableaus and it is designed to preserve primal feasibility on every pivot.

Dictionaries and Simplex Tableaus

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- What is the basic solution identified by T?

Set $x_j = 0 \; j \in N$ so that for $i \in B, \; x_i = (Rb)_r$ if the ith column of T is e_r the rth unit coordinate vector (or, equivalently, the rth column of the identity matrix).

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• When is T optimal? $Rb \geq 0$, $c^T - y^T A \leq 0$ and $-y \leq 0$, i.e. it is primal-dual feasible.

• When is T primal degenerate? $Rb \geq 0$ and $\exists\, i$ s.t. $(Rb)_i = 0$.

• When is T dual degenerate? $(c^T - y^T A,\ -y^T) \leq 0$ and $\exists$ a nonbasic i_0 s.t. $(c^T - y^T A,\ -y^T)_{i_0} = 0$.

• When does T show the LP to be unbounded?
$Rb \geq 0$ and $\exists\, j_0$ s.t. $(c^T - y^T A,\ -y^T)_{j_0} > 0$ and the j_0 column of $[RA\ R]$ is non-positive.

Dictionaries and Simplex Tableaus

$$T = \begin{bmatrix} RA & R & Rb \\ c^T - y^T A & -y^T & -y^T b \end{bmatrix}$$

• What is the basic solution identified by T?
Set $x_j = 0$ $j \in N$ so that for $i \in B$, $x_i = (Rb)_r$ if the ith column of T is e_r the rth unit coordinate vector (or, equivalently, the rth column of the identity matrix).
• When is this basic solution a basic feasible solution? $Rb \geq 0$.
• When is T primal feasible? $Rb \geq 0$.
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$Rb \geq 0$ and $\exists\, j_0$ s.t. $(c^T - y^T A,\ -y^T)_{j_0} > 0$ and the j_0 column of $[RA\ R]$ is non-positive.
• When does T show the LP to be infeasible?
Only by showing that the dual is unbounded. That is, T is dual feasible $((c^T - y^T A,\ -y^T)^T \leq 0)$ and $\exists\ i_0$ such that $(Rb)_{i_0} < 0$ with $\hat{a}_{i_0 j} \geq 0$, $j = 1, \ldots, n+m$.

Phase I of the Simplex Algorithm

Consider the following LP in standard form.

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$$\begin{array}{llr} \max & -x_0 & \\ \text{s.t.} & -x_0 + \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \ldots, m \\ & 0 \leq x_j & j = 0, 1, \ldots, n \end{array}$$

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• State the auxiliary problem in matrix form.

$$\begin{array}{ll} \max & \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T \begin{pmatrix} x_0 \\ x \end{pmatrix} \\ \text{s.t.} & [-\mathbf{1}\ A] \begin{pmatrix} x_0 \\ x \end{pmatrix} \leq b \\ & 0 \leq \begin{pmatrix} x_0 \\ x \end{pmatrix} \end{array}$$

(where $\mathbf{1}$ is the vector of all ones.)

Phase I of the Simplex Algorithm

State the initial dictionary for the phase I problem

$$\begin{array}{lll} \max & -x_0 & \\ \text{s.t.} & -x_0 + \sum_{j=1}^{n} a_{ij} x_j \le b_i & i = 1, \dots, m \\ & 0 \le x_j & j = 0, 1, \dots, n \end{array}$$

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$$x_{n+i} = \quad b_i + x_0 - \sum_{j=1}^{n} a_{ij}x_j \qquad i = 1, \ldots, m$$

$$w = \quad -x_0$$

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Is the initial pivot on this dictionary a standard simplex pivot?

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Is the initial pivot on this dictionary a standard simplex pivot?

NO! The initial pivot is designed to make this dictionary primal feasible so that we can *then apply the primal simplex algorithm* since the primal simplex algorithm requires primal feasibility.

Phase I of the Simplex Algorithm

How does one perform the initial pivot for the initial phase I dictionary

$$\begin{aligned} x_{n+i} &= \quad b_i + x_0 - \sum_{j=1}^{n} a_{ij} x_j \qquad\qquad i = 1, \ldots, m \\ w &= \qquad -x_0 \end{aligned}$$

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x_0 is the entering variable, and the leaving variable is any x_{n+i_0} such that

$$b_{i_0} = \min\{b_i \mid i = 1, \dots, m\} \; < \; 0.$$

Phase I of the Simplex Algorithm

State the initial tableau for the phase I problem

$$\begin{array}{lll} \max & -x_0 & \\ \text{s.t.} & -x_0 + \sum_{j=1}^{n} a_{ij} x_j \le b_i & i = 1, \dots, m \\ & 0 \le x_j & j = 0, 1, \dots, n \end{array}$$

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$$\begin{array}{r} \\ z \text{ row} \\ w \text{ row} \end{array} \left[\begin{array}{rccc|c} -\mathbf{1} & A & I & & b \\ \hline 0 & c^T & 0 & & 0 \\ \hline -1 & 0 & 0 & & 0 \end{array}\right]$$

We have written the objective rows for both the original primal problem and the auxiliary problem in this tableau. In phase I, the z-row (original primal objective row) is just along for the ride so that we can easily initialize phase II.

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We have written the objective rows for both the original primal problem and the auxiliary problem in this tableau. In phase I, the z-row (original primal objective row) is just along for the ride so that we can easily initialize phase II.

How does one perform the initial pivot for the initial phase I tableau?

The first column is the pivot column (i.e. the x_0 column), and the pivot row is any row i_0 for which

$$b_{i_0} = \min \{ b_i \mid i = 1, \ldots, m \} \ < \ 0.$$

Multiple Optimal Solutions

$$\begin{bmatrix} RA & R & Rb \\ c^T - y^T A & -y^T & -y^T b \end{bmatrix}$$

• If this tableau is optimal, how does one detect the existence of multiple primal optimal solutions, and how does one compute them?

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Multiple primal solutions exist if the optimal tableau is dual degenerate. That is, there is a nonbasic variable whose objective row coefficient is zero. Alternative primal optimal BFSs are obtained by performing primal simplex pivots where the pivot column is any one of the dual degenerate columns.

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Multiple dual solutions exist if the optimal tableau is primal degenerate. That is, $\exists\, i_0$ such that $(Rb)_{i_0} = 0$. Alternative dual optimal solutions are obtained by performing dual simplex pivots where the pivot row is any one of the rows i_0 for which $(Rb)_{i_0} = 0$.