FINAL EXAM OUTLINE FOR MATH 407

EXAM DATES:

- Sections AC: Wednesday, December 11, 2019: 8:30 - 10:20am.
- Sections BD: Monday, December 9, 2019: 8:30 - 10:20pm.

EXAM OUTLINE

The final exam will consist of 5 questions each worth 70 points for a total of 350 points. The content of each question is as follows.

Question 1: (Modeling) In this question you will be asked to model one of the LP models 2, 4, 6, 7, 9, 10, 11 12 given on the class web page.

Question 2: (Theory) This question is devoted to the basic theory and concepts studied in this course. The questions and their components are based on the *vocabulary words* that you have studied throughout the quarter. The questions are designed in a manner similar to the theory questions addressed by the midterm exam. In this regard, the questions are designed not only to test rote memory of definitions and theorems but also the underlying concepts and their relationships. In addition, you may be asked to state and prove the Weak Duality Theorem for LPs in either the basic or generalized standard forms. For this, you must use the form of these results as they appear in the course notes provided for this course on the course website and not from any other web source or other LP text.

Question 3: (Solving LPs) In this question you will be given one or more LPs and asked to solve them. The solution method may or may not be specified (graphical, the primal simplex algorithm, the two phase simplex algorithm, the dual simplex algorithm).

You will need to show all of your work to get full credit. In addition, you may be asked to answer a question about the nature of the solution that you have found or the nature of the dual solution, e.g. describe the entire optimal solution set to the primal (dual) problem when more than one optimal solution exists.

Question 4: (LP Techniques) In this question you will be asked to do one or more of the following:

1. Put a given LP into standard form.
2. Formulate the dual of a given LP without first bringing it to standard form.
3. Determine if a given vector solves a given LP using the Complementary Slackness Theorem.
4. Determine if a given vector solves a given LP using the Geometric Duality Theorem.

Question 5: (LP Sensitivity Analysis) In this problem you will be given an LP model, its initial tableau, and an associated optimal tableau. You will then be asked to answer a list sensitivity analysis questions about the problem and it's optimal solution using the techniques developed in the course including the computation of break-even prices, range analysis, and pricing out. Answering these questions properly may require some combination of primal and/or dual simplex pivoting.

SAMPLE QUESTIONS

1. Model the following two problems as LPs.

   (a) A company needs to lease warehouse storage space over the next 5 months. Just how much space will be required in each of these months is known. However, since these space requirements are quite different, it may be most economical to lease only the amount needed each month on a month-by-month basis. On the other hand, the additional cost for leasing space for additional months is much less than for the first month, so it may be less expensive to lease the maximum amount needed for the entire 5 months. Another option is the intermediate approach of changing the total amount of space leased (by adding a new lease and/or having an old lease expire) at least once but not every month.

   The space requirement (in thousands of square feet) and the leasing costs (in hundreds of dollars) for the various leasing periods are as follows:

Month	*Required space*
1	30
2	20
3	40
4	10
5	50

Leasing period (months)	*Cost* ($) *per 1,000 sq ft leased*
1	650
2	1,000
3	1,350
4	1,600
5	1,900

   The objective is to minimize the total leasing cost for meeting the space requirement. Formulate the linear programming model for this problem.

   (b) An electronics company has a contract to deliver 20,000 radios within the next four weeks. The client is willing to pay $20 for each radio delivered by the end of the first week, $18 for those delivered by the end of the second week, $16 by the end of the third week, and $14 by the end of the fourth week. Since each worker can assemble only 50 radios per week, the company cannot meet the order with its present labor force of 40; hence it must hire and train temporary help. Any of the experienced workers can be taken off the assembly line to instruct a class of three trainees; after one week of instruction, each of the trainees can either proceed to the assembly line or instruct additional new classes.

   At present, the company has no other contracts; hence some workers may become idle once the delivery is completed. All of them, whether permanent or temporary, must be kept on the payroll 'til the end of the fourth week. The weekly wages of a worker, whether assembling, instructing, or being idle, are $200; the weekly wages of a trainee are $100. The production costs, excluding the worker's wages, are $5 per radio. The company's aim is to maximize the total net profit. Formulate this problem as an LP problem.

   (c) (Personnel Scheduling) At the beginning of the fall semester, the university computer facility needs to assign working hours to the TAs. Because all the TAs are currently enrolled in the university, the main concern is to assign the work schedule so as not to interfere with class and study times. There are six TAs, all having different wage rates based on expertise and experience. The following table shows their wage rates, along with the maximum number of hours each can work each day.

Operator	*Wage Rate*	*Maximum hours of availability*				
		Mon.	Tue.	Wen.	Thu.	Fri.
K.C	\$6.00/hr	6	0	6	0	6
D.H.	\$6.10/hr	0	6	0	6	0
H.B.	\$5.90/hr	4	8	4	0	4
S.C.	\$5.80/hr	5	5	5	0	5
K.S.	\$6.80/hr	3	0	3	8	0
N.K.	\$7.30/hr	0	0	0	6	2

The goal is to minimize costs while maintaining knowledge of the operation and growing experienced workers. For this reason, it is determined that the less experienced TAs work at least 8 hours per week while the experienced TAs (K.S., N.K.) work at least 7 hours a week.

The computer facility is to be open for operation from 8 a.m. to 10 p.m. Monday through Friday with exactly one operator on duty during these hours. On Saturdays and Sundays, the computer facility is to be operated by other staff.

Formulate a linear programming model to determine the number of hours to be assigned to each TA on each day.

(d) (Investing Over Time) An investor has money-making activities A and B available at the beginning of each of the next 5 years. Each dollar invested in A at the beginning of 1 year returns \$1.40 (a profit of \$0.40) 2 years later (in time for immediate re-investment). Each dollar invested in B at the beginning of 1 year returns \$1.70 3 years later.

In addition, investments C and D will each be available at one time in the future. Each dollar investment in C at the beginning of year 2 returns \$1.90 at the end of year 5. Each dollar invested in D at the beginning of year 5 returns \$1.30 at the end of year 5. Money uninvested in a given year earns 3% per annum.

The investor begins with \$50,000 and wishes to know which investment plan maximizes cash balance at the beginning of year 6. Formulate the linear programming model for this problem.

(e) A cargo plane has three compartments for storing cargo: front, center, and back. These compartments have capacity limits on both *weight* and *space*, as summarized below:

Compartment	*Weight capacity (tons)*	*Space capacity (cu ft)*
Front	12	7,000
Center	18	9,000
Back	10	5,000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane.

The following four cargoes have been offered for shipment on an upcoming flight as space is available:

Cargo	*Weight (tons)*	*Volume (cu ft/ton)*	*Profit (\$/ton)*
1	20	500	280
2	16	700	360
3	25	600	320
4	13	400	250

Any portion of these cargoes can be accepted. The object is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight. Model this problem as an LP.

2. The following questions refer to an LP in standard form¿

$$\mathcal{P} \qquad \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b,\ 0 \leq x\ . \end{array}$$

(a) What does it mean to say that $\mathcal{P}$ is unbounded? Provide and example of an unbounded LP.

(b) What is the initial dictionary for this LP?

(c) What is the initial tableau for this LP?

(d) Provide the definition for a general dictionary for this LP by writing it as a system of equations and stating the necessary and sufficient conditions taht this system must satisfy. The label this dictionary as (D).

(e) When is (D) primal feasible?

(f) When is (D) dual feasible?

(g) When is (D) primal degenerate?

(h) When is (D) dual degenerate?

(i) When is (D) optimal?

(j) When does (D) indicate that $\mathcal{P}$ is unbounded?

(k) If (D) is optimal, under what conditions does (D) indicate that there are multiple optimal primal solutions?

(l) If (D) is optimal, under what conditions does (D) indicate that there are multiple optimal dual solutions?

(m) What is the relationship between dictionaries and simplex tableaus for LPs in standard form?

(n) What is the dual for $\mathcal{P}$?

(o) Give the block matrix equation that illustrates the transition from the initial tableau to any intermediary tableau. Denote the intermediary tableau as T_k, where k stands for the kth tableau.

(p) Under what conditions is this initial tableau T_0 primal feasible?

(q) Under what conditions is T_k primal feasible?

(r) Under what conditions is T_k dual feasible?

(s) Under what conditions is T_k primal degenerate?

(t) Under what conditions is T_k dual degenerate?

(u) Under what conditions is T_k optimal?

(v) If T_k is optimal, under what conditions does T_k indicate that there are multiple optimal primal solutions?

(w) If T_k is optimal, under what conditions does T_k indicate that there are multiple optimal dual solutions?

(x) If T_k is optimal and it indicates that there are multiple optimal primal solutions, how does one compute the other optimal optimal primal solutions?

(y) If T_k is optimal and it indicates that there are multiple optimal dual solutions, how does one compute the other optimal optimal dual solutions?

(z) State the auxiliary problem for an LP in standard form and explain what it is used for. In addition, show that the auxiliary problem is always feasible and bounded.

(i) Describe the first pivot applied to the tableau for the auxiliary problem.

(ii) Describe the four possible outcomes of the two phase simplex algorithm.

(iii) Give a geometric description of the constraint region for a linear program.

(iv) Give a geometric description of what the simplex algorithm is doing.

(v) Give a geometric description of degeneracy.

3. Use the Simplex Algorithm to solve the following LPs stating their solution, the solution to their duals, and their optimal values. (Solution methods other than the simplex algorithm will be given zero credit)

(a)

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & 4x_1 & + & 4x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 40 \\
 & x_1 & + & x_2 & + & 2x_3 & + & x_4 & \leq & 40 \\
 & 2x_2 & + & 2x_2 & + & 3x_3 & + & x_4 & \leq & 60 \\
 & 3x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & \leq & 50 \\
 & 0 & \leq & x_1, & & x_2, & & x_3, & & x_4.
\end{array}$$

(b)

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & -2x_1 & - & 2x_2 & - & x_3 & - & 5x_4 & & \\
\text{subject to} & 2x_1 & - & x_2 & + & x_3 & - & x_4 & \leq & 4 \\
 & & & x_2 & + & 2x_3 & - & x_4 & \leq & 5 \\
 & x_1 & - & x_2 & - & x_3 & - & x_4 & \leq & -3 \\
 & 0 & \leq & x_1, & & x_2, & & x_3, & & x_4.
\end{array}$$

(c)

$$\begin{array}{lrcrcrcr}
\text{maximize} & x_1 & + & x_2 & + & 3x_3 & & \\
\text{subject to} & x_1 & - & x_2 & - & 2x_3 & \leq & -2 \\
 & x_1 & + & 2x_2 & + & 2x_3 & \leq & 2 \\
 & 0 & \leq & x_1, & & x_2, & & x_3.
\end{array}$$

(d)

$$\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & 4x_2 & & \\
\text{subject to} & x_1 & + & x_2 & \leq & 1 \\
 & x_1 & - & x_2 & \leq & -2 \\
 & 0 & \leq & x_1, & & x_2.
\end{array}$$

4. (a) Put the following LP in standard form.

$$\begin{array}{lrcrcrcr}
\text{minimize} & & - & x_2 & + & x_3 & & \\
\text{subject to} & x_1 & & & - & 4x_3 & \geq & -5 \\
 & -3x_1 & + & x_2 & & & = & -3 \\
 & x_1 & + & x_2 & + & x_3 & \leq & 10 \\
 & x_1 \geq & -1 & & , & 0 \geq & x_2 &
\end{array}$$

(b) Formulate a dual for the following LPs.

i.

$$\begin{array}{lrcl}
\text{minimize} & c^T x & & \\
\text{subject to} & Ax & \leq & 0 \\
 & Bx & = & 0,
\end{array}$$

where $c \in \mathbb{R}^n,\ A \in \mathbb{R}^{s\times n}$, and $B \in \mathbb{R}^{t\times n}$.

ii.

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 10x_3 \\ \text{subject to} & x_1 + x_2 - x_3 = 12 \\ & x_1 - x_2 + x_3 \leq 8 \\ & 0 \leq x_2 \leq 10 \end{array}$$

(c) Use *both* the Complementary Slackness Theorem and the Geometric Duality Theorem to determine if the vector $x = (0, 5, 0, 1, 1)^T$ solves the LP

$$\begin{array}{lrcrcrcrcrcl} \text{maximize} & & & x_2 & & & + & 5x_4 & + & 5x_5 & & \\ \text{subject to} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & & & \leq & 11 \\ & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & + & x_5 & \leq & 10 \\ & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & \leq & -2 \\ & x_1 & & & & & + & x_4 & + & 3x_5 & \leq & 4 \\ & 0 & \leq & x_1, & & x_2, & & x_3, & & x_4, & & x_5 \end{array}$$

5. (a) (Silicon Chip Corp) A Silicon Valley firm specializes in making four types of silicon chips for personal computers. Each chip must go through four stages of processing before completion. First the basic silicon wafers are manufactured, second the wafers are laser etched with a micro circuit, next the circuit is laminated onto the chip, and finally the chip is tested and packaged for shipping. The production manager desires to maximize profits during the next month. During the next 30 days she has enough raw material to produce 4000 silicon wafers. Moreover, she has 600 hours of etching time, 900 hours of lamination time, and 700 hours of testing time. Taking into account depreciated capital investment, maintenance costs, and the cost of labor, each raw silicon wafer is worth \$1, each hour of etching time costs \$40, each hour of lamination time costs \$60, and each hour of inspection time costs \$10. The production manager has formulated her problem as a linear program with the following initial tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

where x_1, x_2, x_3, x_4 represent the number of 100 chip batches of the four types of chips. After solving by the Simplex Algorithm, the final tableau is:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
0.5	1	0	0	.015	0	0	−.05	25
−5	0	0	0	−.05	1	0	−.5	50
0	0	1	0	−.02	0	.1	0	10
0.5	0	0	1	.015	0	−.1	.05	5
−1500	0	0	0	−5	0	−100	−50	−145,000

Answer each of the following questions as if it were a separate event. Do not consider the cumulative effects between problems.

i. A new product is to be considered for production. This chip requires 25 hours of etching, 20 hours lamination, and 20 hours testing time per 100 chip batch. What is the breakeven sale price for this chip, that is, at what sale price does it become efficient to introduce this product into the optimal production schedule?

ii. A competitor has just come out with a chip that serves the same market as our type 4 chip. A price war is imminent. By how much can we reduce the profitability of this chip and yet have it remain in the optimal production schedule?

iii. A flu is going around and has hit the shop pretty bad, causing an across the board 10% reduction in the etching, lamination, and testing time. What effect will this have on the optimal production schedule and profitability?

(b) Concrete Products Corporation has the capability of producing four types of concrete blocks. Each block must be subjected to four processes: batch mixing, mold vibrating, inspection, and yard drying. The plant manager desires to maximize profits during the next month. During the upcoming 30 days, he has 800 machine hours available on the batch mixer, 1000 hours on the mold vibrator, and 340 man-hours of inspection time. Yard-drying time is unconstrained. Taking into consideration depreciated capital investment and maintenance costs, batch mixing time is worth \$5 per hour, mold vibrating time is worth \$10 per hour, and inspection time is worth \$10 per hour, and the materials costs for the blocks are \$50, \$80, \$100, and \$120 per pallet, respectively. The production director has formulated his problem as a linear program with the following initial tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
batch mixing	1	2	10	16	1	0	0	800
mold vibrating	1.5	2	4	5	0	1	0	1000
inspection	0.5	0.6	1	2	0	0	1	340
	80	140	300	500	0	0	0	0

where x_1, x_2, x_3, x_4 represent the number of pallets of the four types of blocks. The cost coefficients in the z-row represent the profit in dollars per pallet (not revenue!!). After solving by the Simplex method, the final tableau is:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
0	1	11	19	1.5	-1	0	200
1	0	-12	-22	-2	2	0	400
0	0	0.4	1.6	0.1	-0.4	1	20
0	0	-280	-400	-50	-20	0	-60000

QUESTIONS FOR THE CONCRETE PRODUCTION PROBLEM

Answer each of the following questions as if it were a separate event. Do not consider the cumulative effects between problems.

i. How much must a pallet of type 3 blocks be sold for in order to make it efficient to produce them?

ii. What is the minimum price at which type 2 blocks can be sold and and yet maintain them in the optimal production mix?

iii. If the 800 machine hours on the batch mixer is uncertain, for what range of hours of batch mixing time is it efficient for the optimal production mix to consist of type 1 and 2 blocks?

iv. A competitor has offered the manager additional batch mixing time at \$30 an hour. Neglecting transportation costs, should the manager accept this offer and if so, how many hours of batch mixing time should he purchase at this price?

v. The market for type 2 blocks has gotten hot lately. We can now sell them for \$30 more than we used to. If we make this increase, what is the new optimal production schedule ?

vi. The mold vibrator needs major repairs. Consequently, we will lose 300 hours of mold vibrating time this month. What should be the new production schedule for this month ?

vii. We intend to introduce a new type of block. This block requires 4 hours of batch mixing time, 4 hours mold vibrating time, and 1 hour of inspection time per pallet. The materials costs for this type of block are $80 per pallet. At what price must this product be sold in order to make it efficient to produce ?