FINAL EXAM SAMPLE PROBLEM PARTIAL SOLUTIONS FOR MATH 407

1. All these models have been done in class or are on course webpages.

2. Use the Simplex Algorithm to solve the following LPs stating their solution, the solution to their duals, and their optimal values. (Solution methods other than the simplex algorithm will be given zero credit)

   (a)
$$\begin{array}{lrcrcrcrcr}
\text{maximize} & 4x_1 & + & 4x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & \le & 40 \\
 & x_1 & + & x_2 & + & 2x_3 & + & x_4 & \le & 40 \\
 & 2x_2 & + & 2x_2 & + & 3x_3 & + & x_4 & \le & 60 \\
 & 3x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & \le & 50 \\
 & 0 & \le & x_1, & & x_2, & & x_3, & & x_4.
\end{array}$$

   Solution: $x = (0,\ 15, 10,\ 0)^T$ (or $x = (0, 5, 15, 5)^T$), $y = (0, 0, 1, 1)^T$, $z = 110$. So the optimal solution set is given by
$$\{(1-\lambda)(0,\ 15, 10,\ 0)^T + \lambda(0, 5, 15, 5)^T \,|\, 0 \le \lambda \le 1\}.$$

   (b)
$$\begin{array}{lrcrcrcrcr}
\text{maximize} & -2x_1 & - & 2x_2 & - & x_3 & - & 5x_4 & & \\
\text{subject to} & 2x_1 & - & x_2 & + & x_3 & - & x_4 & \le & 4 \\
 & & & x_2 & + & 2x_3 & - & x_4 & \le & 5 \\
 & x_1 & - & x_2 & - & x_3 & - & x_4 & \le & -3 \\
 & 0 & \le & x_1, & & x_2, & & x_3, & & x_4.
\end{array}$$

   Solution: $x = (0, 1, 2, 0)^T,\ y = (0, 1, 3)^T,\ z = -4.$

   (c)
$$\begin{array}{lrcrcrcr}
\text{maximize} & x_1 & + & x_2 & + & 3x_3 & & \\
\text{subject to} & x_1 & - & x_2 & - & 2x_3 & \le & -2 \\
 & x_1 & + & 2x_2 & + & 2x_3 & \le & 2 \\
 & 0 & \le & x_1, & & x_2, & & x_3.
\end{array}$$

   Solution: $x = (0, 0, 1)^T,\ y = (0,\ 3/2)^T + t(1, 1)^T$ for $t \ge 0$ and, $z = 3$.

   (d)
$$\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & 4x_2 & & \\
\text{subject to} & x_1 & + & x_2 & \le & 1 \\
 & x_1 & - & x_2 & \le & -2 \\
 & 0 & \le & x_1, & & x_2.
\end{array}$$

   Solution: The optimal solution to the auxiliary problem is $-1/2$ and is given by $(x_0, x_1, x_3)^T = (1/2, 0, 3/2)^T$. Hence the original LP is infeasible.

3. (a) Put the following LP in standard form.
$$\begin{array}{lrcrcrcr}
\text{minimize} & & - & x_2 & + & x_3 & & \\
\text{subject to} & x_1 & & & - & 4x_3 & \ge & -5 \\
 & -3x_1 & + & x_2 & & & = & -3 \\
 & x_1 & + & x_2 & + & x_3 & \le & 10 \\
 & x_1 \ge & -1 & , & 0 \ge & x_2 & &
\end{array}$$

   Solution: $x_1 = z_1 - 1,\ x_2 = -z_2,\ x_3 = z_3^+ - z_3^-$
$$\begin{array}{lrcrcrcrcr}
\text{maximize} & & - & z_2 & - & z_3^+ & + & z_3^- & & \\
\text{subject to} & -z_1 & & & + & 4z_3^+ & - & 4z_3^- & \le & 4 \\
 & -3z_1 & - & z_2 & & & & & \le & -6 \\
 & 3z_1 & + & z_2 & & & & & \le & 6 \\
 & z_1 & - & z_2 & + & z_3^+ & - & z_3^- & \le & 11 \\
 & 0 & \le & z_1, & & z_2, & & z_3^+, & & z_3^-
\end{array}$$

(b) Formulate a dual for the following LPs.

i.

$$\begin{array}{lrcl} \text{minimize} & c^T x & & \\ \text{subject to} & Ax & \leq & 0 \\ & Bx & = & 0, \end{array}$$

where $c \in I\!R^n$, $A \in I\!R^{s\times n}$, and $B \in I\!R^{t\times n}$.

Solution:

$$\begin{aligned} &\max\ 0 \\ &\text{ s.t. } A^T u + B^T v = -c \\ &\qquad 0 \leq u\ . \end{aligned}$$

ii.

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 10x_3 \\ \text{subject to} & x_1 + x_2 - x_3 = 12 \\ & x_1 - x_2 + x_3 \leq 8 \\ & 0 \leq x_2 \leq 10 \end{array}$$

Solution:

$$\begin{aligned} &\min\ 12y_1 + 8y_2 + 10y_3 \\ &\text{ s.t. } y_1 + y_2 = 2 \\ &\qquad y_1 - y_2 + y_3 \geq -3 \\ &\qquad -y_1 + y_2 = 10 \\ &\qquad 0 \leq y_2, y_3\ . \end{aligned}$$

4. Use both the Complementary Slackness Theorem and the Geometric Duality Theorem to determine if the vector $x = (0, 5, 0, 1, 1)^T$ solves the LP

$$\begin{array}{lrcrcrcrcrcr} \text{maximize} & & & x_2 & & & + & 5x_4 & + & 5x_5 & & \\ \text{subject to} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & & & \leq & 11 \\ & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & + & x_5 & \leq & 10 \\ & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & \leq & -2 \\ & x_1 & & & & & + & x_4 & + & 3x_5 & \leq & 4 \\ & 0 & \leq & x_1, & & x_2, & & x_3, & & x_4, & & x_5 \end{array}$$

Solution: The given x is not optimal.

Complementary slackness: The associated linear system that needs to be solved to establish this is given by

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 5 \\ 0 \end{pmatrix},$$

which gives $y_4 = -3/5 < 0$.

Geometric duality: The associated linear system that needs to be solved to establish this is given by

$$\begin{bmatrix} -1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix} \begin{pmatrix} r_1 \\ r_3 \\ y_1 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 5 \\ 5 \end{pmatrix},$$

which also gives $y_4 = -3/5$.

5. Solutions to sensitivity problems available on course webpage.