FINAL EXAM OUTLINE FOR MATH 407

EXAM DATES:

- Section A: Monday, December 12, 2015: 8:30 -10:20am.
- Section B: Wednesday, December 14, 2015: 2:30 -4:20pm.

EXAM OUTLINE

The final exam will consist of 6 questions each worth 60 points (except for problem 2 which is worth 50 points) for a total of 350 points. The content of each question is as follows.

Question 1: (Modeling) In this question you will be asked to model one or more of the LP models 1–25 given on the class web page.

Question 2: (Theory) In this question you will be asked to state (*not prove*) one or more of the following key results from this course for an LP in standard form:

1. The weak duality theorem.
2. The fundamental theorem of linear programming.
3. The strong duality theorem.
4. The Fundamental Theorem of Sensitivity Analysis.

In addition, you may be asked to state one or more of the following two theorems.

5. The Fundamental Theorem of the Alternative for a matrix $A \in \mathbb{R}^{m\times n}$.
6. The Existence and Uniqueness Theorem for the Linear Least Squares Problem.

The statements you give for these results **must** come from the the document devoted to to these results appearing on the course home page. Statements for these results, that differ from the texts, can be found elsewhere; however, I require that you use the ones from the online course notes (see the special listing of these results on the course homepage). These are the only *vocabulary words* you will need to know for the final.

Question 3: (Solving LPs) In this question you will be given one or more LPs and asked to solve them. The solution method may or may not be specified (graphical, the primal simplex algorithm, the two phase simplex algorithm, the dual simplex algorithm).

You will need to show all of your work to get full credit. In addition, you may be asked to answer a question about the nature of the solution that you have found or the nature of the dual solution, e.g. describe the entire optimal solution set to the primal (dual) problem when more than one optimal solution exists.

Question 4: (LP Techniques) In this question you will be asked to do one or more of the following:

1. Put a given LP into standard form.
2. Formulate the dual of a given LP without first bringing it to standard form.
3. Determine if a given vector solves a given LP using the Complementary Slackness Theorem.
4. Determine if a given vector solves a given LP using the Geometric Duality Theorem.

Question 5: (LP Sensitivity Analysis) In this problem you will be given an LP model, its initial tableau, and an associated optimal tableau. You will then be asked to answer a list sensitivity analysis questions about the problem and it's optimal solution using the techniques developed in the course including the computation of break-even prices, range analysis, and pricing out. Answering these questions properly may require some combination of primal and/or dual simplex pivoting.

Question 6: (Linear Least Squares and Least Distance to an Affine Set) This problem will have two parts. The first part will be a theoretical question concerning the linear least squares problem and the second will be computational. In the theory question you will be asked to establish properties of the underlying linear algebraic structures associated with the normal equations and related orthogonal projections as well as the solution set. In addition, you may be asked to formulate an underlying least distance problem for and affine set (an *affine set* is any set of the form $\hat{x} + S$ where $\hat{x} \in \mathbb{R}^n$ and S is a subspace of $\mathbb{R}^n$) and to discuss the nature of the underlying solution using projection matrices. In the computational question you may be asked to compute a solution to a linear least squares problem, a QR factorization, or an orthogonal projection, and to use these objects to answer associated questions.

SAMPLE QUESTIONS

1. Model the following two problems as LPs.

   (a) A company needs to lease warehouse storage space over the next 5 months. Just how much space will be required in each of these months is known. However, since these space requirements are quite different, it may be most economical to lease only the amount needed each month on a month-by-month basis. On the other hand, the additional cost for leasing space for additional months is much less than for the first month, so it may be less expensive to lease the maximum amount needed for the entire 5 months. Another option is the intermediate approach of changing the total amount of space leased (by adding a new lease and/or having an old lease expire) at least once but not every month.

   The space requirement (in thousands of square feet) and the leasing costs (in hundreds of dollars) for the various leasing periods are as follows:

Month	*Required space*	*Leasing period (months)*	*Cost (\$) per 1,000 sq ft leased*
1	30	1	650
2	20	2	1,000
3	40	3	1,350
4	10	4	1,600
5	50	5	1,900

   The objective is to minimize the total leasing cost for meeting the space requirement. Formulate the linear programming model for this problem.

   (b) An electronics company has a contract to deliver 20,000 radios within the next four weeks. The client is willing to pay \$20 for each radio delivered by the end of the first week, \$18 for those delivered by the end of the second week, \$16 by the end of the third week, and \$14 by the end of the fourth week. Since each worker can assemble only 50 radios per week, the company cannot meet the order with its present labor force of 40; hence it must hire and train temporary help. Any of the experienced workers can be taken off the assembly line to instruct a class of three trainees; after one week of instruction, each of the trainees can either proceed to the assembly line or instruct additional new classes.

   At present, the company has no other contracts; hence some workers may become idle once the delivery is completed. All of them, whether permanent or temporary, must be kept on the payroll 'til the end of the fourth week. The weekly wages of a worker, whether assembling, instructing, or being idle, are \$200; the weekly wages of a trainee are \$100. The production costs, excluding the worker's wages, are \$5 per radio. The company's aim is to maximize the total net profit. Formulate this problem as an LP problem.

2. State the standard form for an LP used in this class for the development of the simplex algorithm, then state all of the following results for LPs having this standard form:

   (a) The weak duality theorem.

   (b) The fundamental theorem of linear programming.

   (c) The strong duality theorem.

   (d) The Fundamental Theorem of Sensitivity Analysis.

   Moreover, state the Existence and Uniqueness Theorem for the Linear Least Squares Problem.

3. (a) Solve the following LP stating its solution and optimal value.

$$\begin{array}{lrcrcrcrcl} \text{maximize} & 4x_1 & + & 4x_2 & + & 5x_3 & + & 3x_4 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 40 \\ & x_1 & + & x_2 & + & 2x_3 & + & x_4 & \leq & 40 \\ & 2x_2 & + & 2x_2 & + & 3x_3 & + & x_4 & \leq & 60 \\ & 3x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & \leq & 50 \\ & 0 & \leq & x_1, & & x_2, & & x_3, & x_4. & \end{array}$$

(b) State the dual of this LP and give its solution.

4. (a) Put the following LP in standard form.

$$\begin{array}{lrcrcrcl} \text{minimize} & & - & x_2 & + & x_3 & & \\ \text{subject to} & x_1 & & & - & 4x_3 & \geq & -5 \\ & -3x_1 & + & x_2 & & & = & -3 \\ & x_1 & + & x_2 & + & x_3 & \leq & 10 \\ & x_1 \geq & -1 & , & 0 \geq & x_2 & & \end{array}$$

(b) Formulate a dual for the following LPs.

i.

$$\begin{array}{lrcl} \text{minimize} & c^T x & & \\ \text{subject to} & Ax & \leq & 0 \\ & Bx & = & 0, \end{array}$$

where $c \in I\!R^n$, $A \in I\!R^{s\times n}$, and $B \in I\!R^{t\times n}$.

ii.

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 10x_3 \\ \text{subject to} & x_1 + x_2 - x_3 = 12 \\ & x_1 - x_2 + x_3 \leq 8 \\ & 0 \leq x_2 \leq 10 \end{array}$$

(c) Use *both* the Complementary Slackness Theorem and the Geometric Duality Theorem to determine if the vector $x = (0, 5, 0, 1, 1)^T$ solves the LP

$$\begin{array}{lrcrcrcrcrcl} \text{maximize} & & & x_2 & & & + & 5x_4 & + & 5x_5 & & \\ \text{subject to} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & & & \leq & 11 \\ & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & + & x_5 & \leq & 10 \\ & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & \leq & -2 \\ & x_1 & & & & & + & x_4 & + & 3x_5 & \leq & 4 \\ & 0 & \leq & x_1, & & x_2, & & x_3, & & x_4, & x_5 & \end{array}$$

5. Concrete Products Corporation has the capability of producing four types of concrete blocks. Each block must be subjected to four processes: batch mixing, mold vibrating, inspection, and yard drying. The plant manager desires to maximize profits during the next month. During the upcoming 30 days, he has 800 machine hours available on the batch mixer, 1000 hours on the mold vibrator, and 340 man-hours of inspection time. Yard-drying time is unconstrained. Taking into consideration depreciated capital investment and maintenance costs, batch mixing time is worth \$5 per hour, mold vibrating time is worth \$10 per hour, and inspection time is worth \$10 per hour, and the materials costs for the blocks are \$50, \$80, \$100, and \$120 per pallet, respectively. The production director has formulated his

problem as a linear program with the following initial tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
batch mixing	1	2	10	16	1	0	0	800
mold vibrating	1.5	2	4	5	0	1	0	1000
inspection	0.5	0.6	1	2	0	0	1	340
	80	140	300	500	0	0	0	0

where x_1, x_2, x_3, x_4 represent the number of pallets of the four types of blocks. The cost coefficients in the z-row represent the profit in dollars per pallet (not revenue!!). After solving by the Simplex method, the final tableau is:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
0	1	11	19	1.5	-1	0	200
1	0	-12	-22	-2	2	0	400
0	0	0.4	1.6	0.1	-0.4	1	20
0	0	-280	-400	-50	-20	0	-60000

QUESTIONS FOR THE CONCRETE PRODUCTION PROBLEM

Answer each of the following questions as if it were a separate event. Do not consider the cumulative effects between problems.

(a) How much must a pallet of type 3 blocks be sold for in order to make it efficient to produce them?

(b) What is the minimum price at which type 2 blocks can be sold and and yet maintain them in the optimal production mix?

(c) If the 800 machine hours on the batch mixer is uncertain, for what range of hours of batch mixing time is it efficient for the optimal production mix to consist of type 1 and 2 blocks?

(d) A competitor has offered the manager additional batch mixing time at \$30 an hour. Neglecting transportation costs, should the manager accept this offer and if so, how many hours of batch mixing time should he purchase at this price?

(e) The market for type 2 blocks has gotten hot lately. We can now sell them for \$30 more than we used to. If we make this increase, what is the new optimal production schedule ?

(f) The mold vibrator needs major repairs. Consequently, we will lose 300 hours of mold vibrating time this month. What should be the new production schedule for this month ?

(g) We intend to introduce a new type of block. This block requires 4 hours of batch mixing time, 4 hours mold vibrating time, and 1 hour of inspection time per pallet. The materials costs for this type of block are \$80 per pallet. At what price must this product be sold in order to make it efficient to produce ?

6. Let $M,\, A \in I\!R^{m\times n}$, $b \in I\!R^m$, $x^0, \hat{x} \in I\!R^n$, and let S be a subspace of $I\!R^n$.

   (a) Theory

   i. If $y^T M x = 0$ for all $y \in I\!R^m$ and $x \in I\!R^n$, show that $M = 0$.

   ii. Show that $\mathrm{Nul}(A^T A) = \mathrm{Nul}(A)$.

   iii. State the Fundamental Theorem of the Alternative for the matrix A and use it and the previous result to show that $\mathrm{Ran}(A^T A) = \mathrm{Ran}(A)$.

   iv. Show that the linear last squares problem

$$\mathcal{LLS} \quad \min_{x\in I\!R^n} \frac{1}{2}\|Ax - b\|_2^2$$

   always has a solution.

v. If $\mathrm{Nul}(A) = \{0\}$, show that the orthogonal projection onto $\mathrm{Ran}(A)$ is given by $P_{\mathrm{Ran}(A)} = A(A^TA)^{-1}A^T$.

vi. If $\mathrm{Ran}(A) = \mathbb{R}^m$, show that the orthogonal projection onto $\mathrm{Nul}(A)$ is given by $P_{\mathrm{Nul}(A)} = I - A^T(AA^T)^{-1}A$.

vii. Suppose $m < n$, $b \in \mathrm{Ran}(A)$, $Ax^0 = b$, and $A\hat{x} \neq b$. Show that the closest point in the set $\{x : Ax = b\}$ to the point $\hat{x}$ is given by

$$\bar{x} := x^0 + P_{\mathrm{Nul}(A)}(\hat{x} - x^0).$$

(b) Computation

i. Let $a \in \mathbb{R}^n \setminus \{0\}$, $\beta \in \mathbb{R}$ and consider the hyperplane $H := \{x : a^Tx = \beta\}$.

A. Show that the orthogonal projector onto $\{a\}^\perp$ is given by $I - \frac{aa^T}{a^Ta}$.

B. Show that the nearest point in the hyperplane H to the origin is $\bar{x} := \frac{\beta}{a^Ta}a$.

C. Suppose $\hat{x} \in \mathbb{R}^n$ is such that $a^T\hat{x} \neq \beta$. Show that the closest point to $\hat{x}$ on the hyperplane H is the point

$$\bar{x} := \hat{x} + \frac{\beta - a^T\hat{x}}{a^Ta}a\ ,$$

and that the distance of $\hat{x}$ to the hyperplane H is $\frac{|\beta - a^T\hat{x}|}{\|a\|_2}$.

D. What is the distance of the point $(-3, 2)^T$ to the line $3x_1 - 2x_2 = 1$, and what is the closest point on the line to this point?

E. What is the distance of the point $(1, 0, 1)^T$ to the plane $x_1 - x_2 + 2x_3 = 3$ and what is the point in the plane that achieves this distance?

F. Given $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, under what conditions do the two hyperplanes $a^Tx = \alpha$ and $b^Tx = \beta$ intersect in a line? If they intersect in a line, what point on this line is the closest point to the origin?

ii. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

A. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.

B. Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.

C. Compute the QR factorization of A.

iii. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

A. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.

B. Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.

C. Compute the QR factorization of A.

iv. Let $a \in \mathbb{R}$ and consider the function

$$f(x_1, x_2, x_3) = \frac{1}{2}[(2x_1 - 2a^4)^2 + (x_1 - x_2)^2 + (ax_2 + x_3)^2 + x_2^2].$$

A. Write this function in the form of the objective function for a linear least squares problem by specifying the matrix A and the vector b.

B. Describe the solution set of this linear least squares problem as a function of a.

v. Find the quadratic polynomial $p(t) = x_0 + x_1 t + x_2 t^2$ that best fits the following data in the least-squares sense:

t	-2	-1	0	1	2
y	2	-10	0	2	1

.

vi. Consider the problem LLS with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

A. What are the normal equations for this A and b.

B. Solve the normal equations to obtain a solution to the problem LLS for this A and b.

C. What is the general reduced QR factorization for this matrix A?

D. Compute the orthogonal projection onto the range of A.

E. Use the recipe

$$\begin{aligned} AP &= Q[R_1\ R_2] && \text{the general reduced QR factorization} \\ \hat{b} &= Q^T b && \text{a matrix-vector product} \\ \bar{w}_1 &= R_1^{-1}\hat{b} && \text{a back solve} \\ \bar{x} &= P \begin{bmatrix} R_1^{-1}\hat{b} \\ 0 \end{bmatrix} && \text{a matrix-vector product.} \end{aligned}$$

to solve LLS for this A and b.

F. If $\bar{x}$ solves LLS for this A and b, what is $A\bar{x} - b$?