3. Solve the following LP stating its solution and optimal value.

$$\begin{array}{lrcrcrcrcl} \text{maximize} & 4x_1 & + & 4x_2 & + & 5x_3 & + & 3x_4 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 40 \\ & x_1 & + & x_2 & + & 2x_3 & + & x_4 & \leq & 40 \\ & 2x_2 & + & 2x_2 & + & 3x_3 & + & x_4 & \leq & 60 \\ & 3x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & \leq & 50 \\ & 0 & \leq & x_1, & & x_2, & & x_3, & & x_4. \end{array}$$

Also, state the dual of this LP and give its solution.

Solution: $x = (0,\ 15, 10,\ 0)^T,\ y = (0, 0, 1, 1)^T,\ z = 110.$

4. (a) Put the following LP in standard form.

$$\begin{array}{lrcrcrcl} \text{minimize} & & - & x_2 & + & x_3 & & \\ \text{subject to} & x_1 & & & - & 4x_3 & \geq & -5 \\ & -3x_1 & + & x_2 & & & = & -3 \\ & x_1 & + & x_2 & + & x_3 & \leq & 10 \\ & x_1 \geq & -1 & , & 0 \geq & x_2 & & \end{array}$$

Solution: $x_1 = z_1 - 1,\ x_2 = -z_2,\ x_3 = z_3^+ - z_3^-$

$$\begin{array}{lrcrcrcrcr} \text{maximize} & & - & z_2 & - & z_3^+ & + & z_3^- & & \\ \text{subject to} & -z_1 & & & + & 4z_3^+ & - & 4z_3^- & \leq & 4 \\ & -3z_1 & - & z_2 & & & & & \leq & -6 \\ & 3z_1 & + & z_2 & & & & & \leq & 6 \\ & z_1 & - & z_2 & + & z_3^+ & - & z_3^- & \leq & 11 \\ & 0 & \leq & z_1, & & z_2, & & z_3^+, & & z_3^- \end{array}$$

(b) Formulate a dual for the following LPs.

i.

$$\begin{array}{lrcl} \text{minimize} & c^T x & & \\ \text{subject to} & Ax & \leq & 0 \\ & Bx & = & 0, \end{array}$$

where $c \in I\!R^n,\ A \in I\!R^{s\times n}$, and $B \in I\!R^{t\times n}$.

Solution:

$$\begin{array}{l} \max\ 0 \\ \text{s.t. } A^T u + B^T v = -c \\ \quad 0 \leq u\ . \end{array}$$

ii.

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 10x_3 \\ \text{subject to} & x_1 + x_2 - x_3 = 12 \\ & x_1 - x_2 + x_3 \leq 8 \\ & 0 \leq x_2 \leq 10 \end{array}$$

Solution:

$$\begin{array}{l} \min\ 12y_1 + 8y_2 + 10y_3 \\ \text{s.t. } y_1 + y_2 = 2 \\ \quad y_1 - y_2 + y_3 \geq -3 \\ \quad -y_1 + y_2 = 10 \\ \quad 0 \leq y_2, y_3\ . \end{array}$$

(c) Use both the Complementary Slackness Theorem and the Geometric Duality Theorem to determine if the vector $x = (0, 5, 0, 1, 1)^T$ solves the LP

$$\begin{array}{lrcrcrcrcrcl} \text{maximize} & & & x_2 & & & + & 5x_4 & + & 5x_5 & & \\ \text{subject to} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & & & \leq & 11 \\ & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & + & x_5 & \leq & 10 \\ & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & \leq & -2 \\ & x_1 & & & & & + & x_4 & + & 3x_5 & \leq & 4 \\ & 0 & \leq & x_1, & & x_2, & & x_3, & & x_4, & & x_5 \end{array}$$

Solution: The given x is not optimal.

6. Let $M,\, A \in I\!R^{m\times n},\ b \in I\!R^m,\ x^0, \hat{x} \in I\!R^n$, and let S be a subspace of $I\!R^n$.

(a) Theory

i. If $y^T M x = 0$ for all $y \in I\!R^m$ and $x \in I\!R^n$, show that $M = 0$.

Solution: $0 = e_i^T M e_j = M_{ij}\ \forall\, i, j$

ii. Show that $\text{Nul}(A^T A) = \text{Nul}(A)$.

Solution: If $Ax = 0$, then $A^T Ax = 0$ so $\text{Nul}(A) \subset \text{Nul}(A^T A)$. On the other hand, if $A^T Ax = 0$, then $0 = x^T A^T Ax = \|Ax\|_2^2$, so $Ax = 0$. Consequently, $\text{Nul}(A^T A) \subset \text{Nul}(A)$, and so these sets are equivalent.

iii. State the Fundamental Theorem of the Alternative (FTA) for the matrix A and use it and the previous result to show that $\text{Ran}(A^T A) = \text{Ran}(A)$.

Solution: The FTA states that, for any matrix $A \in I\!R^{m\times n}$, $\text{Ran}(A) = \text{Nul}(A^T)^\perp$ and $\text{Nul}(A) = \text{Ran}(A^T)^\perp$. By the previous problem $\text{Nul}(A^T A) = \text{Nul}(A)$ and so, by the FTA, $\text{Ran}(A^T A) = \text{Nul}(A^T A)^\perp = \text{Nul}(A)^\perp = \text{Ran}(A^T)$.

iv. Show that the linear last squares problem

$$\mathcal{LLS} \quad \min_{x\in I\!R^n} \frac{1}{2}\|Ax - b\|_2^2$$

always has a solution.

Solution: We know that $\bar{x}$ solves $\mathcal{LLS}$ if and only if $A^T A\bar{x} = A^T b$, and since $\text{Ran}(A^T A) = \text{Ran}(A)$, the system $A^T Ax = A^T b$ must always have a solution. Hence, $\mathcal{LLS}$ must always have a solution.

Note: The hard part here is showing that $\bar{x}$ solves $\mathcal{LLS}$ if and only if $A^T A\bar{x} = A^T b$ which is avoided in this proof.

v. If $\text{Nul}(A) = \{0\}$, show that the orthogonal projection onto $\text{Ran}(A)$ is given by $P_{\text{Ran}(A)} = A(A^T A)^{-1}A^T$.

Solution: First, recall that since $\text{Nul}(A^T A) = \text{Nul}(A)$, we know that $\text{Nul}(A^T A)$ and so the square matrix $A^T A$ is invertible. Hence the matrix $P := A(A^T A)^{-1}A^T$ is well defined. To show that it is the orthogonal projector onto $\text{Ran}(A)$, we need to show that $P^2 = P$, $P^T = P$, and $\text{Ran}(P) = \text{Ran}(A)$. The two facts $P^2 = P$ and $P^T = P$ are obvious upon inspection, so it remains only to show that $\text{Ran}(P) = \text{Ran}(A)$. Obviously, $\text{Ran}(P) \subset \text{Ran}(A)$.
On the other hand, let $y \in \text{Ran}(A)$ so that there exists $x \in I\!R^n$ such that $y = Ax$. Set $z = (A^T A)x$. Then $x = (A^T A)^{-1}z$, and, since the FTA tells us that $\text{Ran}(A^T A) = \text{Ran}(A^T)$, there is a $w \in I\!R^m$ such that $z = A^T w$. Putting this all together gives $y = Ax = A(A^T A)^{-1}z = A(A^T A)^{-1}A^T w = Pw$, and so $\text{Ran}(A) \subset \text{Ran}(P)$ which completes the proof.

vi. If $\text{Ran}(A) = I\!R^m$, show that the orthogonal projection onto $\text{Nul}(A)$ is given by $P_{\text{Nul}(A)} = I - A^T(AA^T)^{-1}A$.

Solution: The FTA tells us that $P_{\text{Nul}(A)} = I - P_{\text{Nul}(A)^\perp} = I - P_{\text{Ran}(A^T)}$, so we need only show that $P_{\text{Ran}(A^T)} = A^T(AA^T)^{-1}A$. By replacing A by A^T in the previous result, we see that $P_{\text{Ran}(A^T)} = A^T(AA^T)^{-1}A$ if $\{0\} = \text{Nul}(A^T)$, or equivalently, $I\!R^m = \text{Nul}(A^T)^\perp = \text{Ran}(A)$ where the final inequality follows from the FTA. This proves the result.

vii. Suppose $m < n$, $b \in \text{Ran}(A)$, $Ax^0 = b$, and $A\hat{x} \neq b$. Show that the closest point in the set $\{x : Ax = b\}$ to the point $\hat{x}$ is given by

$$\bar{x} := x^0 + P_{\text{Nul}(A)}(\hat{x} - x^0).$$

Solution: As previously observed, $\{x : Ax = b\} = x^0 + \text{Nul}(A)$, so we may write this least distance problem as

$$\min_{z \in \text{Nul}(A)} \frac{1}{2} \left\|x^0 + z - \hat{x}\right\|_2^2 = \frac{1}{2} \left\|z - (\hat{x} - x^0)\right\|_2^2$$

whose solution in z is the projection of $(\hat{x} - x^0)$ onto $\text{Nul}(A)$. Therefore, the solution $\bar{x}$ is givan by $\bar{x} = x^0 + P_{\text{Nul}(A)}(\hat{x} - x^0)$.

(b) Computation

i. Let $a \in I\!R^n \setminus \{0\}$, $\beta \in I\!R$ and consider the hyperplane $H := \{x : a^T x = \beta\}$.

A. Show that the orthogonal projector onto $\{a\}^\perp$ is given by $I - \frac{aa^T}{a^Ta}$.

Solution: Set $P := \frac{aa^T}{a^Ta}$. Then $P^2 = P$, $P = P^T$, and $\text{Ran}(P) = \text{Span}\,[a]$. Therefore, $P = P_{\text{Span}\,[a]}$, the orthogonal projection onto the linear span of a. Consequently,

$$I - \frac{aa^T}{a^Ta} = I - P_{\text{Span}\,[a]} = P_{\text{Span}\,[a]^\perp} = P_{\{a\}^\perp}.$$

B. Show that the nearest point in the hyperplane H to the origin is $\bar{x} := \frac{\beta}{a^Ta}a$.

Solution: Note that $H = \left\{\bar{x} + z \,\middle|\, a^T z = 0\right\} = \left\{\bar{x} + z \,\middle|\, z \in \{a\}^\perp\right\}$, so the nearest point in the hyperplane H to the origin solves the problem

$$\min_{z \in \{a\}^\perp} \frac{1}{2} \left\|\bar{x} + z\right\|_2^2.$$

By the previous problem, the solution in z is $\bar{z} = P_{\{a\}^\perp}(-\bar{x}) = (I - \frac{aa^T}{a^Ta})(-\bar{x})$. Hence, the nearest point solution is

$$\bar{x} + (I - \frac{aa^T}{a^Ta})(-\bar{x}) = \frac{aa^T}{a^Ta}\bar{x} = \frac{aa^T}{a^Ta}\frac{\beta}{a^Ta}a = \bar{x}.$$

C. Suppose $\hat{x} \in I\!R^n$ is such that $a^T\hat{x} \neq \beta$. Show that the closest point to $\hat{x}$ on the hyperplane H is the point

$$\bar{x} := \hat{x} + \frac{\beta - a^T\hat{x}}{a^Ta}a\ ,$$

and that the distance of $\hat{x}$ to the hyperplane H is $\frac{|\beta - a^T\hat{x}|}{\|a\|_2}$.

Solution: We follow the reasoning of the previous problem to see that the nearest point in the hyperplane to $\hat{x}$ solves the problem

$$\min_{z\in\{a\}^\perp} \frac{1}{2}\|\bar{x}+z-\hat{x}\|_2^2 = \frac{1}{2}\|z-(\hat{x}-\bar{x})\|_2^2 .$$

The solution in z is given by $\bar{z} = P_{\{a\}^\perp}(\hat{x}-\bar{x})$, an so the solution in x is given by

$$\bar{x}+\bar{z} = \frac{\beta}{a^Ta}a + (I - \frac{aa^T}{a^Ta})(\hat{x}-\bar{x}) = \frac{\beta}{a^Ta}a + (I - \frac{aa^T}{a^Ta})(\hat{x}) = \hat{x} + \frac{\beta - a^T\hat{x}}{a^Ta}a.$$

Hence the distance of $\hat{x}$ to H is

$$\left\|\hat{x} - \left(\hat{x} + \frac{\beta - a^T\hat{x}}{a^Ta}a\right)\right\|_2 = \frac{|\beta - a^T\hat{x}|}{\|a\|_2} .$$

D. What is the distance of the point $(-3,2)^T$ to the line $3x_1 - 2x_2 = 1$, and what is the closest point on the line to this point?

Solution: By the previous problem (with $a = (3,-2)^T$ and $\beta = 1$) the distance is $|1+13|/\sqrt{13} = 14/\sqrt{13}$ an the point on the line that is closest is $\bar{x} = (-3,2)^T + (14/13)(3,-2)^T = (1/13)(3,-2)^T$.

E. What is the distance of the point $(2,0,2)^T$ to the plane $x_1 - x_2 + 2x_3 = 3$ and what is the point in the plane that achieves this distance?

Solution: Just as in the previous problem (with $a = (1,-1,2)^T$ and $\beta = 3$), the distance is $|3-6|/\sqrt{6} = 3/\sqrt{6}$ an the point on the line that is closest is $\bar{x} = (2,0,2)^T - (1/2)(1,-1,2)^T = (3/2,\ 1/2,\ 1)^T$.

ii. Given $a, b \in \mathbb{R}^n \setminus \{0\}$ and $\alpha, \beta \in \mathbb{R}$, under what conditions do the two hyperplanes $H_1 := \left\{x \mid a^Tx = \alpha\right\}$ and $H_2 = \left\{x \mid b^Tx = \beta\right\}$ intersect in a line? If they intersect in a line, what point on this line is the closest point to the origin?

Solution: The intersection of the two hyperplanes is $\{x \mid Ax = c\}$, where

$$A := \begin{bmatrix} a^T \\ b^T \end{bmatrix} \text{ and } c = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

So $A \in \mathbb{R}^{2\times n}$, and the set of solutions (when it is non-empty) is given by $x^0 + \mathrm{Nul}(A)$ where x^0 is any particular solution to $Ax = c$. For the solution to be a line we must have $1 = \mathrm{nullity}(A) \geq n-2$, so it must be the case that $n = 3$ and $\mathrm{rank}(A) = 3-1 = 2$, i.e., a and b are linearly independent.

Recapping, we have shown the intersection of H_1 and H_2 is a line if and only if $n = 3$ and a and b are linearly independent. It remains only to determine the point closed to the origin on this line. For this we need to solve the problem

$$\min_{z\in\mathrm{Nul}(A)} \frac{1}{2}\|x^0+z\|_2^2 ,$$

where x^0 is any particular solution to $Ax = c$. Since $\mathrm{rank}(A) = 2$, AA^T is invertible and $\bar{x} := A^T(AA^T)^{-1}c$ is the least norm solution to $Ax = c$. Written more explicitely, we have

$$\bar{x} = \frac{1}{\|a\|_2^2\|b\|_2^2 - \langle a,\, b\rangle^2}[a^T\ b^T]\begin{bmatrix} \|b\|_2^2 & -\langle a,\, b\rangle \\ -\langle a,\, b\rangle & \|a\|_2^2 \end{bmatrix} c = \xi_1 a^T + \xi_2 b^T,$$

where $c = (c_1,\ c_2)^T$ and

$$\xi_1 = \frac{c_1\|b\|_2^2 - c_2\langle a,\, b\rangle}{\|a\|_2^2\|b\|_2^2 - \langle a,\, b\rangle^2} \text{ and } \xi_2 = \frac{c_1\|a\|_2^2 - c_2\langle a,\, b\rangle}{\|a\|_2^2\|b\|_2^2 - \langle a,\, b\rangle^2}.$$

iii. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

A. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.

Solution: If $AP = QR$ is the QR factorization of A, then the orthogonal projection onto $\mathrm{Ran}(A)$ is given by QQ^T. In class it was shown that

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix},$$

so that

$$P_{\mathrm{Ran}(A)} = QQ^T = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & -1 \\ 1 & -1 & 1 & 3 \end{bmatrix}.$$

B. Compute the orthogonal projection onto $\mathrm{Null}(A)$.

Solution: Since the columns of A are linearly independent we have $\mathrm{Nul}(A) = \{0\}$. Therefor the orthogonal projection on $\mathrm{Nul}(A)$ is given by the 3×3 zero matrix.

C. Compute the QR factorization of A.

Solution: The matrix Q is as given above for A (not the one for A^T) and $P = I$, so we need only compote R knowing that $R_{ij} = \langle A_{\cdot i},\, Q_{\cdot j}\rangle$ giving

$$R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

iv. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

Solution: Follow the recipe given for solving the previous problem starting with the Gram-Schmidt orthogonalization procedure on the columns of A.

A. Compute the orthogonal projection onto $\mathrm{Ran}(A)$.

B. Compute the orthogonal projection onto $\mathrm{Null}(A^T)$.

C. Compute the QR factorization of A.

v. Let $a \in \mathbb{R}$ and consider the function

$$f(x_1, x_2, x_3) = \frac{1}{2}[(2x_1 - 2a^4)^2 + (x_1 - x_2)^2 + (ax_2 + x_3)^2 + x_2^2].$$

A. Write this function in the form of the objective function for a linear least squares problem by specifying the matrix A and the vector b.

Solution:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & a & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } b = \begin{pmatrix} 2a^4 \\ 0 \\ 0 \end{pmatrix}.$$

B. Describe the solution set of this linear least squares problem as a function of a.

Solution: Use the normal equations (or simply differentiate) to show that the unique solution is given by $\bar{x} = (-a^7,\ a^4,\ -a^5,)^T$.

vi. Find the quadratic polynomial $p(t) = x_0 + x_1 t + x_2 t^2$ that best fits the following data in the least-squares sense:

t	-2	-1	0	1	2
y	2	-10	0	2	1

.

Solution: As shown in the notes, we formulate this as a LLS problem with

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \text{ and } b = \begin{pmatrix} 2 \\ -10 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

We then simply solve the normal equations to get the optimal coefficients $(x_0, x_1, x_2)^T$. The normal equations are

$$\begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} x = A^T A x = A^T b = \begin{pmatrix} -5 \\ 10 \\ 4 \end{pmatrix}$$

giving $\bar{x} = (-3,\ 1,\ 1)^T$, or $p(t) = -3 + t + t^2$.

vii. Consider the problem LLS with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Solution: This is essentially the same as problems ii and iii above. Just start with Gram-Schmidt on the columns of A. Of course, one can also solve the problem in a piecemeal fashion by first directly solving the normal equations.

A. What are the normal equations for this A and b.

B. Solve the normal equations to obtain a solution to the problem LLS for this A and b.

C. What is the general reduced QR factorization for this matrix A?

D. Compute the orthogonal projection onto the range of A.

E. Use the recipe

$$\begin{aligned} AP &= Q[R_1\ R_2] && \text{the general reduced QR factorization} \\ \hat{b} &= Q^T b && \text{a matrix-vector product} \\ \bar{w}_1 &= R_1^{-1}\hat{b} && \text{a back solve} \\ \bar{x} &= P\begin{bmatrix} R_1^{-1}\hat{b} \\ 0 \end{bmatrix} && \text{a matrix-vector product.} \end{aligned}$$

to solve LLS for this A and b.

F. If $\bar{x}$ solves LLS for this A and b, what is $A\bar{x} - b$?