SAMPLE QUESTIONS

1.

(a) We first set up some constant vectors for our constraints. Let

$$\begin{aligned} b &= (30, 20, 40, 10, 50)^T, \\ c &= (650, 1000, 1350, 1600, 1900)^T. \end{aligned}$$

Then we set up variables x_{ij}, where $1 \le i, j \le 5$ and $i + j \le 6$. By using x_{ij}, we mean the amount of space (in thousands of square feet) to lease at i-th month for a period of totally j months. Then the LP is

$$\begin{array}{ll} \text{minimize} & \sum_{i,j} c_j x_{ij} \\ \text{subject to} & \sum_{\substack{i \le k \\ i+j \ge k+1}} x_{ij} = b_k \quad \text{for all } 1 \le k \le 5 \\ & 0 \le x_{ij} \end{array}$$

(b) We first set up some constant vectors for our constraints. Let

$$\begin{aligned} c &= (20, 18, 16, 14)^T, \\ d &= (200, 200, 200, 100)^T. \end{aligned}$$

Notice at each week, there are four different types of people at the electronics company

Types	1	2	3	4
	Assemble	Instruct	Idle	Trainees

We use the variables x_{ij} to describe the company's schedule for these four types of people within this four weeks. Here by using x_{ij}, we mean the number of people of type i at j-th week the company has arranged, where $1 \le i, j \le 4$. The LP is

$$\begin{array}{lll} \text{maximize} & \sum_{1 \le j \le 4} 50 x_{1j}(c_j - 5) - \sum_{1 \le i,j \le 4} d_i x_{ij} & \\ \text{subject to} & \sum_{1 \le j \le 4} 50 x_{1j} = 20000 & \\ & x_{2j} \ge 3 x_{4j} & \text{for all } 1 \le j \le 4 \\ & \sum_{1 \le i \le 3} x_{i1} = 40 & \\ & \sum_{1 \le i \le 3} x_{ij} = \sum_{1 \le i \le 4} x_{ij-1} & \text{for all } 2 \le j \le 4 \\ & 0 \le x_{ij} & \end{array}$$

3.

(a)

We apply simplex algorithm to the initial tableau:

$$
\begin{array}{cccccccc|c}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 40 \\
1 & 1 & 2 & 1 & 0 & 1 & 0 & 0 & 40 \\
2 & 2 & 3 & 1 & 0 & 0 & 1 & 0 & 60 \\
3 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 50 \\
\hline
4 & 4 & 5 & 3 & 0 & 0 & 0 & 0 & 0 \\
\hline
-\frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 15 \\
-\frac{1}{2} & 0 & 1 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 15 \\
-1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 10 \\
\frac{3}{2} & 1 & 1 & 1 & 0 & 0 & 0 & \frac{1}{2} & 25 \\
\hline
-2 & 0 & 1 & -1 & 0 & 0 & 0 & -2 & -100 \\
\hline
-\frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 15 \\
\frac{1}{2} & 0 & 0 & 1 & 0 & 1 & -1 & \frac{1}{2} & 5 \\
-1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 10 \\
\frac{5}{2} & 1 & 0 & 2 & 0 & 0 & -1 & \frac{3}{2} & 15 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -110
\end{array}
$$

We end up with an optimal tableau, which is dual generate. Hence we can pivot on the second row to obtain another optimal tableau:

$$
\begin{array}{cccccccc|c}
-\frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 15 \\
\frac{1}{2} & 0 & 0 & 1 & 0 & 1 & -1 & \frac{1}{2} & 5 \\
-\frac{1}{2} & 0 & 1 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 15 \\
\frac{3}{2} & 1 & 0 & 0 & 0 & -2 & 1 & \frac{1}{2} & 5 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -110
\end{array}
$$

We can pivot on the second row, since it is still dual generate. This gives us the first optimal tableau. Therefore the optimal solution is

$$
\left\{ \lambda \begin{pmatrix} 0 \\ 15 \\ 10 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 5 \\ 15 \\ 5 \end{pmatrix} \middle| 0 \le \lambda \le 1 \right\}
$$

.

(b) The dual LP is

$$
\begin{array}{rccccccccc}
\text{minimize} & 40y_1 & + & 40y_2 & + & 60y_3 & + & 50y_4 & & \\
\text{subject to} & y_1 & + & y_2 & + & 2y_3 & + & 3y_4 & \ge & 4 \\
 & y_1 & + & y_2 & + & 2y_3 & + & 2y_4 & \ge & 4 \\
 & y_1 & + & 2y_2 & + & 3y_3 & + & 2y_4 & \ge & 5 \\
 & y_1 & + & y_2 & + & y_3 & + & 2y_4 & \ge & 3 \\
 & 0 & \le & y_1, & & y_2, & & y_3, & y_4 &
\end{array}
$$

The dual solution is $(0,0,1,1)^T$.

4.

(a) We set variables

$$\begin{aligned} x_1 &= z_1 - 1 \\ x_2 &= -z_2 \\ x_3 &= z_3^+ - z_3^- \end{aligned}$$

And we change the LP into the standard form:

$$\begin{array}{lrcrcrcrcr} \text{maximize} & & - & z_2 & + & z_3^+ & - & z_3^- & & \\ \text{subject} & -z_1 & & & + & 4z_3^+ & - & 4z_3^- & \leq & 4 \\ & -3z_1 & - & z_2 & & & & & \leq & -6 \\ & 3z_1 & + & z_2 & & & & & \leq & 6 \\ & z_1 & - & z_2 & + & z_3^+ & - & z_3^- & \leq & 11 \\ & 0 & \leq & z_1, & z_2, & z_3^+, & z_3^- & & & \end{array}$$

(b)
i. Let $u \in \mathbb{R}^s, v \in \mathbb{R}^t$. The dual of the LP is to solve the following linear equations

$$A^T u + B^T v = -c,$$

where $0 \leq u$.
ii. The dual LP is

$$\begin{array}{lrcrcrcr} \text{minimize} & 12y_1 & + & 8y_2 & + & 10y_3 & & \\ & y_1 & + & y_2 & & & = & 2 \\ & y_1 & - & y_2 & + & y_3 & \geq & -3 \\ & -y_1 & + & y_2 & & & = & 10 \\ & 0 & \leq & y_2, & y_3 & & & \end{array}$$

(c)
(i) Use complementary slackness theorem. First plug in the candidate $x = (0, 5, 0, 1, 1)^T$ for the constraints:

$$\begin{array}{rcrcrcrcrcl} 1(0) & + & 2(5) & - & (0) & + & 1(1) & + & 0(1) & = & 11 \\ 3(0) & + & 1(5) & + & 4(0) & + & 1(1) & + & 1(1) & < & 10 \\ 2(0) & - & 1(5) & + & 2(0) & + & 1(1) & + & 2(1) & = & -2 \\ 1(0) & + & 0(5) & + & 0(0) & + & 1(1) & + & 3(1) & = & 4 \end{array}$$

By the Corollary 1.1(i), $y_2 = 0$. Moreover since $x_2, x_4, x_5 \neq 0$, we have by Corollary 1.1 (ii)

$$\begin{aligned} 2y_1 + y_2 - y_3 &= 1 \\ y_1 + y_2 + y_3 + y_4 &= 5 \\ y_2 + 2y_3 + 3y_3 &= 5 \\ y_2 &= 0 \end{aligned}$$

We form the associated augmented matrix

$$\left|\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 3 & 5 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right|$$

By solving it, we have the dual solution $y = \frac{1}{5}(11, 0, 17, -3)$. Certainly it is NOT feasible for the dual LP. Hence by complementary slackness theorem, $x = (0, 5, 0, 1, 1)^T$ is not the optimal solution for the primal.

(ii) Use Geometric Duality theorem. The active hyperplanes are

$$\begin{array}{ccccccccccc} x_1 & + & 2x_2 & - & x_3 & + & x_4 & & & = & 11 \\ 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & = & -2 \\ x_1 & & & & & + & x_4 & + & 3x_5 & = & 4 \\ -x_1 & & & & & & & & & = & 0 \\ & & & - & x_3 & & & & & = & 0 \end{array}$$

This means that $y_2 = 0$ and y_1, y_3, y_4 are obtained by solving

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_3 \\ y_4 \\ r_1 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

We form the associated augmented matrix

$$\left|\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 5 \\ 0 & 2 & 3 & 0 & 0 & 5 \end{array}\right|$$

By solving it, we have

$$\begin{pmatrix} y_1 \\ y_3 \\ y_4 \\ r_1 \\ r_3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 11 \\ 17 \\ -3 \\ 42 \\ 23 \end{pmatrix}$$

Since $y_4 < 0$, the the geometric duality theorem, we know $x = (0, 5, 0, 1, 1)^T$ is NOT the optimal solution for the primal.

5. a) We need to compute the break even price for type (3) shirt, since in the current optimal tableau we do not produce any of them. First we compute it in the unit of a batch

of shirts (20 shirts).

$$\begin{aligned}\text{current sale price} &= \text{current profit} + \text{costs}\\ &= \text{current profit} + \text{shirt cost} + \text{paint cost} + \text{labor cost} + \text{dye cost}\\ &= \$200 + \$7 \times 20 + \$0.2 \times 2 \times 20 + \$0.75 \times 1 \times 20\\ &= \$363\end{aligned}$$

Then the break even price of one batch of type (3) shirts is

$$\begin{aligned}\text{break even price} &= \text{current sale price} + \text{reduce cost}\\ &= \$363 + \$40\\ &= \$403\end{aligned}$$

Then the break even sale price for one single shirt of type (3) should be $\$403/20 = 21.50$.

b) In this part, we need to price out a new product, which has $a_{\text{new}} = (1, 0, 3, 3)^T$. The profit for one batch of this new product is

$$\begin{aligned}\text{current profit} &= \text{current sale price} - \text{costs}\\ &= \text{current sale price} - \text{shirt cost} - \text{paint cost} - \text{labor cost} - \text{dye cost}\\ &= \$19.25 \times 20 - \$7 \times 20 - \$0.75 \times 3 \times 20\\ &= \$200\end{aligned}$$

Denote R as the record matrix, we have

$$Ra_{\text{new}} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{5}{2} & -2 & 1 & -\frac{3}{2} \\ 3 & -1 & 0 & -1 \\ -\frac{3}{2} & 1 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

And $c_{\text{new}} - a_{\text{new}}^T y = 200 - (1, 0, 3, 3)(30, 80, 0, 50)^T = 20$. Then we have the following tableau

$$\begin{array}{ccccccccc|c} 1 & 1 & 0 & 0 & 0 & -.5 & 0 & 0 & 0.5 & 5 \\ 1 & 0 & 0 & -1 & 0 & 2.5 & -2 & 1 & -1.5 & 5 \\ 0 & 0 & 1 & 0 & 0 & 3 & -1 & 0 & -1 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1.5 & 1 & 0 & 0.5 & 10 \\ \hline 20 & 0 & 0 & -40 & 0 & -30 & -80 & 0 & -50 & -41,00 \end{array}$$

It is not dual feasible, hence it is not optimal. We can do a simple pivot on the first row to recover its optimality.

$$\begin{array}{ccccccccc|c} 1 & 1 & 0 & 0 & 0 & -.5 & 0 & 0 & 0.5 & 5 \\ 0 & -1 & 0 & -1 & 0 & 3 & -2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & -1 & 0 & -1 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1.5 & 1 & 0 & 0.5 & 10 \\ \hline 0 & -20 & 0 & -40 & 0 & -20 & -80 & 0 & -40 & -42,00 \end{array}$$

Hence the new production schedule is to produce the new type 5 batches, type (2) 5 batches, type (4) 10 batches. And we do not product any type (1) or type (3) shirts.

(c) We need to do a range analysis of two resources, paints and dyes. Say we increase the amount of paints by θ and dyes by ϕ. Suppose we do not want to change the current bases for the optimal solution, then we have

$$\begin{pmatrix} 5 \\ 5 \\ 5 \\ 10 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ -1 \\ 1 \end{pmatrix} \theta + \begin{pmatrix} .5 \\ -1.5 \\ -1 \\ 0.5 \end{pmatrix} \phi \geq 0$$

It becomes a LP problem

$$\begin{array}{rccc} \text{maximize} & 80\theta + 50\phi & + & 4100 \\ \text{subject to} & 2\theta + 1.5\phi & \leq & 5 \\ & \theta + \phi & \leq & 5 \\ & 0 & \leq & \theta, \phi \end{array}$$

It is easy to see the optimal solution is $\theta = 5/2, \phi = 0$. If $\theta \geq 5/2$, we can do a dual simplex pivot on the second row of the current optimal tableau. This will bring the shadow price of dyes to zero. But the shadow price of the paints will go down to $\$80 - \$50 \times 2 \times 2/3 = \$13.\bar{3}$. This means that Arty prefers to buy as many paints as possible to increase the total profit. The most Arty is willing to pay for each paint: (i) when she buy less than 2.5 batches, then the price is $\$0.2 + \$80/20 = \$4.2$; (ii) for more than 2.5 batches, the most she is wiling to pay is $\$0.2 + \$13.\bar{3}/20 = \$0.8\bar{6}$.