Math 407 — Linear Optimization

1 Introduction

1.1 What is optimization?

A mathematical optimization problem is one in which some function is either maximized or minimized relative to a given set of alternatives. The function to be minimized or maximized is called the *objective function* and the set of alternatives is called the feasible region (or constraint region). In this course, the feasible region is always taken to be a subset of $\mathbb{R}^n$ (real n-dimensional space) and the objective function is a function from $\mathbb{R}^n$ to $\mathbb{R}$.

We further restrict the class of optimization problems that we consider to linear programming problems (or LPs). An LP is an optimization problem over $\mathbb{R}^n$ wherein the objective function is a linear function, that is, the objective has the form

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

for some $c_i \in \mathbb{R}$ $i = 1, \ldots, n$, and the feasible region is the set of solutions to a finite number of linear inequality and equality constraints, of the form

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \qquad i = 1, \ldots, s$$

and

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \qquad i = s+1, \ldots, m.$$

Linear programming is an extremely powerful tool for addressing a wide range of applied optimization problems. A short list of application areas is resource allocation, production scheduling, warehousing, layout, transportation scheduling, facility location, flight crew scheduling, portfolio optimization, parameter estimation, ...

1.2 An Example

To illustrate some of the basic features of LP, we begin with a simple two-dimensional example. In modeling this example, we will review the four basic steps in the development of an LP model:

1. Identify and label the *decision variables*.
2. Determine the objective and use the decision variables to write an expression for the *objective function* as a linear function of the decision variables.
3. Determine the *explicit constraints* and write a functional expression for each of them as either a linear equation or a linear inequality in the decision variables.

4. Determine the *implicit constraints*, and write each as either a linear equation or a linear inequality in the decision variables.

   PLASTIC CUP FACTORY

   A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is \$25 while the profit on a case of champagne glasses is \$20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

We will model the problem of maximizing the profit for this company as an LP. The first step in our modeling process is to identify and label the *decision variables*. These are the variables that represent the quantifiable decisions that must be made in order to determine the daily production schedule. That is, we need to specify those quantities whose values completely determine a production schedule and its associated profit. In order to determine these quantities, one can ask the question "If I were the plant manager for this factory, what must I know in order to implement a production schedule?" The best way to identify the decision variables is to put oneself in the shoes of the decision maker and then ask the question "What do I need to know in order to make this thing work?" In the case of the plastic cup factory, everything is determined once it is known how many cases of beer mugs and champagne glasses are to be produced each day.

Decision Variables:

$B = \#$ of cases of beer mugs to be produced daily.

$C = \#$ of cases of champagne glasses to be produced daily.

You will soon discover that the most difficult part of any modeling problem is identifying the decision variables. Once these variables are correctly identifies then the remainder of the modeling process usually goes smoothly.

After identifying and labeling the decision variables, one then specifies the problem objective. That is, write an expression for the objective function as a linear function of the decision variables.

Objective Function:

Maximize profit where profit $= 25B + 20C$

The next step in the modeling process is to express the feasible region as the solution set of a finite collection of linear inequality and equality constraints. We separate this process into two steps:

1. determine the explicit constraints, and
2. determine the implicit constraints.

The explicit constraints are those that are explicitly given in the problem statement. In the problem under consideration, there are explicit constraints on the amount of resin and the number of work hours that are available on a daily basis.

Explicit Constraints:

resin constraint: $20B + 12C \leq 1800$

work hours constraint: $\frac{1}{15}B + \frac{1}{15}C \leq 8$.

This problem also has other constraints called implicit constraints. These are constraints that are not explicitly given in the problem statement but are present nonetheless. Typically these constraints are associated with "natural" or "common sense" restrictions on the decision variable. In the cup factory problem it is clear that one cannot have negative cases of beer mugs and champagne glasses. That is, both B and C must be non-negative quantities.

Implicit Constraints:

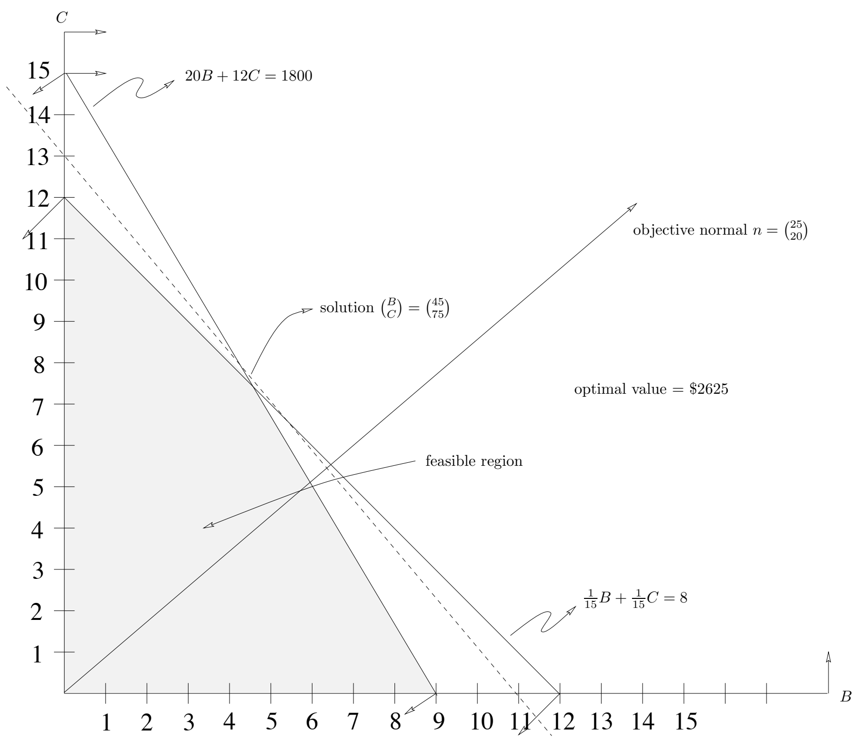
$$0 \leq B, \qquad 0 \leq C.$$

The entire model for the cup factory problem can now be succinctly stated as

$$\begin{array}{rlrcl} \mathcal{P} : & \max 25B + 20C & & & \\ & \text{subject to} & 20B + 12C & \leq & 1800 \\ & & \frac{1}{15}B + \frac{1}{15}C & \leq & 8 \\ & & 0 & \leq & B, C \end{array}$$

Since it is an introductory example, the Plastic Cup Factory problem is particularly easy to model. As the course progresses you will be asked to model problems of increasing difficulty and complexity. In this regard, let me emphasize again that the first step in the modeling process, identification of the decision variables, is always the most difficult. In addition, the 4 step modeling process outlined above is not intended to be a process that one steps through in a linear fashion. As the model unfolds it is often necessary to revisit earlier steps, for example by adding in more decision variables (a very common requirement). Moving between these steps several times is often required before the model is complete. In this process, the greatest stumbling block experienced by students is the overwhelming desire to try to solve the problem as it is being modeled. Indeed, every student who has taken this course over has made this error (and on occasion I continue to make this error myself). Perhaps the most common error in this regard is to try to reduce the total number of decision variables required. This often complicates the modeling process, blocks the ability to fully characterize all of the variability present, makes it difficult to interpret the solution and understand its robustness, and makes it difficult to modify the model as it evolves. **Never be afraid to add more decision variables** either to clarify the model or to improve its flexibility. Modern LP software easily solves problems with tens of thousands of variables, and in some cases tens of millions of variables. It is more important to get a correct, easily interpretable, and flexible model then to provide a compact minimalist model.

We now turn to solving the Plastic Cup Factory problem. Since this problem is two dimensional it is possible to provide a graphical representation and solution. The first step is to graph the feasible region. To do this, first graph



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the line associated with each of the linear inequality constraints. Then determine on which side of each of these lines the feasible region must lie (don't forget the implicit constraints!). To determine the correct side, locate a point not on the line that determines the constraint (for example, the origin is often not on the line, and it is particularly easy to use). Plug this point in and see if it satisfies the constraint. If it does, then it is on the correct side of the line. If it does not, then the other side of the line is correct. Once the correct side is determined put little arrows on the line to remind yourself of the correct side. Then shade in the resulting feasible region which is the set of points feasible for all of the linear inequalities.

The next step is to draw in the vector representing the gradient of the objective function. This vector may be placed anywhere on your graph, but it is often convenient to draw it emanating from the origin. Since the objective function has the form

$$f(x_1, x_2) = c_1 x_1 + c_2 x_2,$$

the gradient of f is the same at every point in $\mathbb{R}^2$;

$$\nabla f(x_1, x_2) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Recall from calculus that the gradient always points in the direction of increasing function values. Moreover, since the gradient is constant on the whole space, the level sets of f associated with different function values are given by the lines perpendicular to the gradient. Consequently, to obtain the location of the point at which the objective is maximized we simply set a ruler perpendicular to the gradient and then move the ruler in the direction of the gradient until we reach the last point (or points) at which the line determined by the ruler intersects the feasible region. In the case of the cup factory problem this gives the solution to the LP as $\binom{B}{C} = \binom{45}{75}$

We now recap the steps followed in the solution procedure given above:

Step 1: Graph each of the linear constraints indicating on which side of the constraint the feasible region must lie with an arrow. Don't forget the implicit constraints!

Step 2: Shade in the feasible region.

Step 3: Draw the gradient vector of the objective function.

Step 4: Place a straight-edge perpendicular to the gradient vector and move the straight-edge either in the direction of the gradient vector for maximization (or in the opposite direction of the gradient vector for minimization) to the last point for which the straight-edge intersects the feasible region. The set of points of intersection between the straight-edge and the feasible region is the set of solutions to the LP.

Step 5: Compute the exact optimal vertex solutions to the LP as the points of intersection of the lines on the boundary of the feasible region indicated in Step 4. Then compute the resulting optimal value associated with these points.

The solution procedure described above for two dimensional problems reveals a great deal about the geometric structure of LPs that remains true in n dimensions. We will explore this geometric structure more fully as the course evolves. For the moment, note that the solution to the Plastic Cup Factory problem lies at a *corner point* of the feasible region. Indeed, it is easy to convince oneself that every 2 dimensional LP has an optimal solution that is such a *corner point*. The notion of a corner point can be generalized to n dimensional space where it is referred to as a *vertex*. These vertices play a big role in understanding the geometry of linear programming.

Before leaving this section, we make a final comment on the modeling process described above. We emphasize that there is not one and only one way to model the Cup Factory problem, or any problem for that matter. In particular, there are many ways to choose the decision variables for this problem. Clearly, it is sufficient for the shop manager to know how many hours each day should be devoted to the manufacture of beer mugs and how many hours to champagne glasses. From this information everything else can be determined. For example, the number of cases of beer mugs that get produced is 15 times the number of hours devoted to the production of beer mugs. However, in the end, they should all yield the same optimal process.

1.3 Sensitivity Analysis

One of the most important things to keep in mind about "real world" LPs is that the input data associated with the problem specification can change over time, is subject to measurement error, and is often the product of educated guesses (another name for fudging). For example, in the case of the cup factory the profit levels for both beer mugs and champagne glasses are subject to seasonal variations. Prior to the New Year, the higher demand for champagne glasses forces up the sale price and consequently their profitability. As St. Patrick's Day approaches the demand for champagne glasses drops, but the demand for beer mugs soars. In June, demand for champagne glasses again rises due to the increase in marriage celebrations. Then, just before the Fourth of July, the demand for beer mugs returns. These seasonal fluctuations may effect the optimal solution and the optimal value. Similarly, the availability of the resources required to produce the beer mugs and champagne glasses as well as their purchase prices vary with time, as well as changes and innovations in the market place. In this context, it is natural, indeed, often essential, to ask how the optimal value and optimal solutions change as the input data for the problem changes. The mathematical study of these changes is called *sensitivity analysis*. This is a very important area of linear programming. Although we reserve our detailed study of this topic to the end of the course, it is useful to introduce some of these ideas now to motivate several important topics in linear programming. The most important of these being *duality theory*. We begin with the optimal value function and marginal values.

1.3.1 The Optimal Value Function and Marginal Values

Next consider the effect of fluctuations in the availability of resources on both the optimal solution and the optimal value. In the case of the cup factory there are two basic resources consumed by the production process: plastic resin and labor hours. In order to analyze the behavior of the problem as the value of these resources is perturbed, we first observe a geometric property of the optimal solution, namely that the optimal solution lies at a "corner point" or "vertex" of the feasible region. More will be made of the notion of a vertex later, but for the moment suffice it to say that if an optimal solution to an LP exists then there is at least one optimal solution that is a vertex of the feasible region. Next note that as the availability of a resource is changed the constraint line associated with that resource moves in a parallel fashion along a line normal to the constraint. Thus, at least for a small range of perturbations to the resources, the vertex associated with the current optimal solution moves but remains optimal. (We caution that this is only a generic property of an optimal vertex and there are examples for which it fails; for example, in some models the feasible region can be made empty under arbitrarily small perturbations of the resources.) These observations lead us to conjecture that the solution to the LPs

$$\begin{array}{rcl} v(\epsilon_1, \epsilon_2) & = & \max 25B + 20C \\ & & \text{subject to } \begin{array}{rcl} 20B + 12C & \leq & 1800 + \epsilon_1 \\ \frac{1}{15}B + \frac{1}{15}C & \leq & 8 + \epsilon_2 \\ 0 & \leq & B, C \end{array} \end{array}$$

lies at the intersection of the two lines $20B + 12C = 1800 + \epsilon_1$ and $\frac{1}{15}B + \frac{1}{15}C = 8 + \epsilon_2$ for small values of ϵ_1 and ϵ_2; namely

$$\begin{array}{rcl} B & = & 45 - \frac{45}{2}\epsilon_2 + \frac{1}{8}\epsilon_1 \\ C & = & 75 + \frac{75}{2}\epsilon_2 - \frac{1}{8}\epsilon_1, \text{ and} \\ v(\epsilon_1, \epsilon_2) & = & 2625 + \frac{375}{2}\epsilon_2 + \frac{5}{8}\epsilon_1. \end{array}$$

It can be verified by direct computation that this indeed yields the optimal solution for small values of ϵ_1 and ϵ_2.

Next observe that the value $v(\epsilon_1, \epsilon_2)$ can now be viewed as a function of ϵ_1 and ϵ_2 and that this function is differentiable at $\binom{\epsilon_1}{\epsilon_2} = \binom{0}{0}$ with

$$\nabla v(\epsilon_1, \epsilon_2) = \left[\begin{array}{c} 5/8 \\ 375/2 \end{array} \right].$$

The number $\frac{5}{8}$ is called the marginal value of the resin resource at the optimal solution $\binom{B}{C} = \binom{45}{75}$, and the number $\frac{375}{2}$ is called the marginal value of the labor time resource at the optimal solution $\binom{B}{C} = \binom{45}{75}$. We have the following interpretation for these marginal values: each additional pound of resin beyond the base amount of 1800 lbs. contributes $\$\frac{5}{8}$ to the profit and each additional hour of labor beyond the base amount of 8 hours contributes $\$\frac{375}{2}$ to the profit.

Using this information one can answer certain questions concerning how one might change current operating limitations. For example, if we can buy additional resin from another supplier, how much more per pound are we willing to pay than we are currently paying? (Answer: $\$\frac{5}{8}$ per pound is the most we are willing to pay beyond what we now pay, why?) Or, if we are willing to add overtime hours, what is the greatest overtime salary we are willing to pay? Of course, the marginal values are only good for a certain range of fluctuation in the resources, but within that range they provide valuable information.

1.4 Duality Theory

We now briefly discuss the economic theory behind the marginal values and how the "hidden hand of the market place" gives rise to them. This leads in a natural way to a mathematical theory of duality for linear programming.

Think of the cup factory production process as a black box through which the resources flow. Raw resources go in one end and exit the other. When they come out the resources have a different form, but whatever comes out is still comprised of the entering resources. However, something has happened to the value of the resources by passing through the black box. The resources have been purchased for one price as they enter the box and are sold in their new form when they leave. The difference between the entering and exiting prices is called the profit. Assuming that there is a positive profit the resources have increased in value as they pass through the production process.

Let us now consider how the market introduces pressures on the profitability and the value of the resources available to the market place. We take the perspective of the cup factory *vs* the market place. The market place does not want the cup factory to go out of business. On the other hand, it does not want the cup factory to see a profit. It wants to keep all the profit for itself and only let the cup factory just break even. It does this by setting the price of the resources available in the market place. That is, the market sets the price for plastic resin and labor and it tries to do so in such a way that the cup factory sees no profit and just breaks even. Since the cup factory is now seeing a profit, the market must figure out by how much the sale price of resin and labor must be raised to reduce this profit to zero. This is done by minimizing the value of the available resources over all price increments that guarantee that the cup factory either loses money or sees no profit from both of its products. If we denote the per unit price increment for resin by R and that for labor by L, then the profit for beer mugs is eliminated as long as

$$20R + \frac{1}{15}L \geq 25$$

since the left hand side represents the increased value of the resources consumed in the production of one case of beer mugs and the right hand side is the current profit on a case of beer mugs. Similarly, for champagne glasses, the market wants to choose R and L so that

$$12R + \frac{1}{15}L \geq 20.$$

Now in order to maintain equilibrium in the market place, that is, not drive the cup factory out of business (since then the market realizes no profit at all), the market chooses R and L so as to minimize the increased value of the available resources. That is, the market chooses R and L to solve the problem

$$\begin{array}{rlrcl} \mathcal{D} & : & \text{minimize}\, 1800R + 8L & & \\ & & \text{subject to} \;\; 20R + \frac{1}{15}L & \geq & 25 \\ & & 12R + \frac{1}{15}L & \geq & 20 \\ & & 0 & \leq & R, L \end{array}$$

This is just another LP. It is called the "dual" to the LP $\mathcal{P}$ in which the cup factory tries to maximize profit. Observe that if $\binom{B}{C}$ is feasible for $\mathcal{P}$ and $\binom{R}{L}$ is feasible for $\mathcal{D}$, then

$$\begin{array}{rcl} 25B + 20C & \leq & [20R + \frac{1}{15}L]B + [12R + \frac{1}{15}L]C \\ & = & R[20B + 12C] + L[\frac{1}{15}B + \frac{1}{15}C] \\ & \leq & 1800R + 8L. \end{array}$$

Thus, the value of the objective in $\mathcal{P}$ at a feasible point in $\mathcal{P}$ is bounded above by the objective in $\mathcal{D}$ at any feasible point for $\mathcal{D}$. In particular, the optimal value in $\mathcal{P}$ is bounded above by the optimal value in $\mathcal{D}$. The "strong duality theorem" states that if either of these problems has a finite optimal value, then so does the other and these values coincide. In addition, we claim that the solution to $\mathcal{D}$ is given by the marginal values for $\mathcal{P}$. That is, $\binom{R}{L} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$ is the optimal solution for $\mathcal{D}$. In order to show this we need only show that $\binom{R}{L} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$ is feasible for $\mathcal{D}$ and that the value of the objective in $\mathcal{D}$ at $\binom{R}{L} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$ coincides with the value of the objective in $\mathcal{P}$ at $\binom{B}{C} = \binom{45}{75}$. First we check feasibility:

$$0 \leq \frac{5}{8}, \qquad 0 \leq \frac{375}{2}$$

$$20 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 25$$

$$12 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \geq 20.$$

Next we check optimality

$$25 \cdot 45 + 20 \cdot 75 = 2625 = 1800 \cdot \frac{5}{8} + 8 \cdot \frac{375}{2}.$$

This is a most remarkable relationship! We have shown that the marginal values have three distinct and seemingly disparate interpretations:

1. The marginal values are the partial derivatives of the value function for the LP with respect to resource availability,

2. The marginal values give the per unit increase in value of each of the resources that occurs as a result of the production process, and

3. The marginal values are the solutions to a dual LP, $\mathcal{D}$.

1.5 LPs in Standard Form and Their Duals

Recall that a linear program is a problem of maximization or minimization of a linear function subject to a finite number of linear inequality and equality constraints. This general definition leads to an enormous variety of possible formulations. In this section we propose one fixed formulation for the purposes of developing an algorithmic solution procedure and developing the theory of linear programming. We will show that every LP can be recast in this one fixed form. We say that an LP is in *standard form* if it has the form

$$\begin{array}{lll} \mathcal{P}: & \text{maximize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ & \text{subject to} & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \ \leq \ b_i \quad \text{for } i = 1, 2, \ldots, m \\ & & 0 \leq x_j \quad \text{for } j = 1, 2, \ldots, n\ . \end{array}$$

Using matrix notation, we can rewrite this LP as

$$\begin{array}{lll} \mathcal{P}: & \text{maximize} & c^Tx \\ & \text{subject to} & Ax \leq b \\ & & \quad 0 \leq x\ , \end{array}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and the inequalities $Ax \leq b$ and $0 \leq x$ are to be interpreted componentwise.

Following the results of the previous section on LP duality, we claim that the dual LP to $\mathcal{P}$ is the LP

$$\begin{array}{lll} \mathcal{D}: & \text{minimize} & b_1y_1 + b_2y_2 + \cdots + b_my_m \\ & \text{subject to} & a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \ \geq \ c_j \text{ for } j = 1, 2, \ldots, n \\ & & 0 \leq y_i \text{ for } i = 1, 2, \ldots, m\ , \end{array}$$

or, equivalently, using matrix notation we have

$$\begin{array}{lll} \mathcal{D}: & \text{minimize} & b^Ty \\ & \text{subject to} & A^Ty \geq c \\ & & 0 \leq y\ . \end{array}$$

Just as for the cup factory problem, the LPs $\mathcal{P}$ and $\mathcal{D}$ are related via the *Weak Duality Theorem* for linear programming.

Theorem 1.1 (Weak Duality Theorem) *If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then*

$$c^T x \le y^T A x \le b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

PROOF: Let $x \in \mathbb{R}^n$ be feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ be feasible for $\mathcal{D}$. Then

$$\begin{array}{rcll}
c^T x & = & \sum_{j=1}^{n} c_j x_j & \\
 & \le & \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j & [\text{since } 0 \le x_j \text{ and } c_j \le \sum_{i=1}^{m} a_{ij} y_i, \text{ so } c_j x_j \le (\sum_{i=1}^{m} a_{ij} y_i) x_j] \\
 & = & y^T A x & \\
 & = & \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i & \\
 & \le & \sum_{i=1}^{m} b_i y_i & [\text{since } 0 \le y_i \text{ and } \sum_{j=1}^{n} a_{ij} x_j \le b_i, \text{ so } (\sum_{j=1}^{n} a_{ij} x_j) y_i \le b_i y_i] \\
 & = & b^T y &
\end{array}$$

To see that $c^T \bar{x} = b^T \bar{y}$ plus $\mathcal{P}$–$\mathcal{D}$ feasibility implies optimality, simply observe that for every other $\mathcal{P}$–$\mathcal{D}$ feasible pair (x, y) we have

$$c^T x \le b^T \bar{y} = c^T \bar{x} \le b^T y \ .$$

■

We caution that the infeasibility of either $\mathcal{P}$ or $\mathcal{D}$ does not imply the unboundedness of the other. Indeed, it is possible for both $\mathcal{P}$ and $\mathcal{D}$ to be infeasible as is illustrated by the following example.

EXAMPLE:

$$\begin{array}{rrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
 & x_1 & - & x_2 & \le & 1 \\
 & -x_1 & + & x_2 & \le & -2 \\
 & 0 & \le & x_1, & x_2 &
\end{array}$$

1.5.1 Transformation to Standard Form

Every LP can be transformed to an LP in standard form. This process usually requires a transformation of variables and occasionally the addition of new variables. In this section we provide a step-by-step procedure for transforming any LP to one in standard form.

minimization → maximization

To transform a minimization problem to a maximization problem just multiply the objective function by -1.

linear inequalities

If an LP has an equality constraint of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i,$$

it can be transformed to one in standard form by multiplying the inequality through by -1 to get

$$-a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \leq -b_i.$$

linear equation

The linear equation

$$a_{i1}x_i + \cdots + a_{in}x_n = b_i$$

can be written as two linear inequalities

$$a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$$

and

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i.$$

The second of these inequalities can be transformed to standard form by multiplying through by -1.

variables with lower bounds

If a variable x_i has lower bound l_i which is not zero ($l_i \leq x_i$), one obtains a non-negative variable w_i with the substitution

$$x_i = w_i + l_i.$$

In this case, the bound $l_i \leq x_i$ is equivalent to the bound $0 \leq w_i$.

variables with upper bounds

If a variable x_i has an upper bound u_i ($x_i \leq u_i$) one obtains a non-negative variable w_i with the substitution

$$x_i = u_i - w_i.$$

In this case, the bound $x_i \leq u_i$ is equivalent to the bound $0 \leq w_i$.

variables with interval bounds

An interval bound of the form $l_i \le x_i \le u_i$ can be transformed into one non-negativity constraint and one linear inequality constraint in standard form by making the substitution

$$x_i = w_i + l_i.$$

In this case, the bounds $l_i \le x_i \le u_i$ are equivalent to the constraints

$$0 \le w_i \qquad \text{and} \qquad w_i \le u_i - l_i.$$

free variables

Sometimes a variable is given without any bounds. Such variables are called free variables. To obtain standard form every free variable must be replaced by the difference of two non-negative variables. That is, if x_i is free, then we get

$$x_i = u_i - v_i$$

with $0 \le u_i$ and $0 \le v_i$.

To illustrate the ideas given above, we put the following LP into standard form.

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 3x_1 & - & x_2 & & & & & & \\
\text{subject to} & -x_1 & + & 6x_2 & - & x_3 & + & x_4 & \ge & -3 \\
 & & & 7x_2 & & & + & x_4 & = & 5 \\
 & & & & & x_3 & + & x_4 & \le & 2
\end{array}$$

$$-1 \le x_2, x_3 \le 5, -2 \le x_4 \le 2.$$

The hardest part of the translation to standard form, or at least the part most susceptible to error, is the replacement of existing variables with non-negative variables. For this reason, I usually make the translation in two steps. In the first step I make all of the changes that do not involve variable substitution, and then, in the second step, I start again and do all of the variable substitutions. Following this procedure, let us start with all of the transformations that do not require variable substitution. First, turn the minimization problem into a maximization problem by rewriting the objective as

$$\text{maximize } -3x_1 + x_2.$$

Next we replace the first inequality constraint by the constraint

$$x_1 - 6x_2 + x_3 - x_4 \le 3.$$

The equality constraint is replaced by the two inequality constraints

$$\begin{array}{rcrcr}
7x_2 & + & x_4 & \le & 5 \\
-7x_2 & - & x_4 & \le & -5.
\end{array}$$

Finally, the double bound $-2 \leq x_4 \leq 2$ indicates that we should group the upper bound with the linear inequalities. All of these changes give the LP

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & -3x_1 & + & x_2 & & & & & & \\
\text{subject to} & x_1 & - & 6x_2 & + & x_3 & - & x_4 & \leq & 3 \\
& & & 7x_2 & & & + & x_4 & \leq & 5 \\
& & - & 7x_2 & & & - & x_4 & \leq & -5 \\
& & & & & x_3 & + & x_4 & \leq & 2 \\
& & & & & & & x_4 & \leq & 2
\end{array}$$

$$-1 \leq x_2, x_3 \leq 5, -2 \leq x_4.$$

We now move on to variable replacement. Observe that the variable x_1 is free, so we replace it by

$$x_1 = z_1^+ - z_1^- \text{ with } 0 \leq z_1^+, 0 \leq z_1^-.$$

The variable x_2 has a non-zero lower bound so we replace it by

$$z_2 = x_2 + 1 \quad \text{or} \quad x_2 = z_2 - 1 \quad \text{with} \quad 0 \leq z_2.$$

The variable x_3 is bounded above, so we replace it by

$$z_3 = 5 - x_3 \quad \text{or} \quad x_3 = 5 - z_3 \quad \text{with} \quad 0 \leq z_3.$$

The variable x_5 is bounded below, so we replace it by

$$z_4 = x_4 + 2 \quad \text{or} \quad x_4 = z_4 - 2 \quad \text{with} \quad 0 \leq z_4.$$

After making these substitutions, we get the following LP in standard form:

$$\begin{array}{lrcrcrcrcrcr}
\text{maximize} & -3z_1^+ & + & 3z_1^- & + & z_2 & & & & & & \\
\text{subject to} & z_1^+ & - & z_1^- & - & 6z_2 & - & z_3 & - & z_4 & \leq & -10 \\
& & & & & 7z_2 & & & + & z_4 & \leq & 14 \\
& & & & - & 7z_2 & & & - & z_4 & \leq & -14 \\
& & & & & & - & z_3 & + & z_4 & \leq & -1 \\
& & & & & & & & & z_4 & \leq & 4
\end{array}$$

$$0 \leq z_1^+, z_1^-, z_2, z_3, z_4.$$