THE FAMILY FARM

Fred and Martha Schmertz work a 40 acre family farm outside of Athens Georgia. They grow tubers. Specifically, they grow potatoes, yams, beets, and turnips. Planting season is coming around and they must decide how many acres of each type of tuber to grow. The resources required to plant, cultivate, and harvest these tubers are acres, fertilizer, machine hours, and person hours. The annual property tax is $100 per acre and this tax is levied only if an acre is put into production. Fertilizer costs $10 per 100 lbs and they have 4000 lbs on order. Although several types of machines are used in planting and cultivation, they can all be rented from the local co-op at $20 per hour up to their total machine time allocation which is 600 hours. The labor costs are $5 per hour and Martha estimates that they will be able to obtain up to 500 hours of labor this season. Fred and Martha ask the co-op agent for advice on what and how much to plant. The agent decides to model their problem as a linear program and obtains the following initial tableau:

	P	Y	B	T	S_1	S_2	S_3	S_4	b
acres	1	1	1	1	1	0	0	0	40
fertilizer	100	100	200	100	0	1	0	0	4000
machinetime	20	20	30	10	0	0	1	0	600
labor	30	20	20	20	0	0	0	1	500
	400	400	500	300	0	0	0	0	0

where the cost coefficients in the z-row is the profit per acre in dollars. After applying the simplex algorithm the agent obtained the following optimal tableau:

P	Y	B	T	S_1	S_2	S_3	S_4	b
50	0	0	100	0	1	-10	5	500
2.5	1	0	2	0	0	$-.1$	.15	15
$-.5$	0	0	0	1	0	0	$-.05$	15
-1	0	1	-1	0	0	.1	$-.1$	10
-100	0	0	0	0	0	-10	-10	-11000

1. Are there any other planting strategies that Fred and Martha should consider before implementing the one described in the optimal tableau given below? If so, describe them, that is compute the entire new tableau associated with any other viable planting strategy.

Solution

Turnips are non-basic in the given optimal tableau with a reduced cost of zero. Hence there are multiple optimal solutions. In particular, turnips can be made basic in an alternative optimal tableau. Below we pivot on the turnip column to obtain this alternative optimal tableau.

P	Y	B	T	S_1	S_2	S_3	S_4	b
50	0	0	**100**	0	1	-10	5	500
2.5	1	0	2	0	0	$-.1$	.15	15
$-.5$	0	0	0	1	0	0	$-.05$	15
-1	0	1	-1	0	0	.1	$-.1$	10
-100	0	0	0	0	0	-10	-10	-11000
.5	0	0	1	0	.01	$-.1$	.05	5
1.5	1	0	0	0	$-.02$	.1	.05	5
$-.5$	0	0	0	1	0	0	$-.05$	15
$-.5$	0	1	0	0	.01	0	$-.05$	15
-100	0	0	0	0	0	-10	-10	-11000

The alternative basic optimal solution is

$$(P, Y, B, T) = (0, 5, 15, 5)\ .$$

The set of all optimal solutions is

$$\left\{ \lambda \begin{pmatrix} 0 \\ 15 \\ 10 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 5 \\ 15 \\ 5 \end{pmatrix} \middle| \, 0 \le \lambda \le 1 \right\}\ .$$

2. What is the break even sale price of an acre of potatoes? If the market has changed so that we can now sell an acre of potatoes for \$1185, what is the new planting schedule?

Initial Tableau:

	P	Y	B	T	S_1	S_2	S_3	S_4	b
acres	1	1	1	1	1	0	0	0	40
fertilizer	100	100	200	100	0	1	0	0	4000
machinetime	20	20	30	10	0	0	1	0	600
labor	30	20	20	20	0	0	0	1	500
	400	400	500	300	0	0	0	0	0

Optimal Tableaus:

P	Y	B	T	S_1	S_2	S_3	S_4	b
50	0	0	<u>100</u>	0	1	-10	5	500
2.5	1	0	2	0	0	$-.1$	.15	15
$-.5$	0	0	0	1	0	0	$-.05$	15
-1	0	1	-1	0	0	.1	$-.1$	10
-100	0	0	0	0	0	-10	-10	-11000
.5	0	0	1	0	.01	$-.1$	.05	5
1.5	1	0	0	0	$-.02$	.1	.05	5
$-.5$	0	0	0	1	0	0	$-.05$	15
$-.5$	0	1	0	0	.01	0	$-.05$	15
-100	0	0	0	0	0	-10	-10	-11000

Solution

a) The break–even sale price for potatoes is given by

$$\text{costs+profit+reduces cost}$$
$$= [100 + 10 + 400 + 150] + [400] + [100] = \$1160 \ .$$

b) The new cost row coefficient for potatoes is

$$1185 - 1160 = 25.$$

Thus, potatoes come into production. The second row is the pivot row. Multiplying this row by 2/5 yields the row

$$[\,1 \quad 2/5 \quad 0 \quad 4/5 \quad 0 \quad 0 \quad -1/25 \quad 3/50 \;|\; 6\,]\,.$$

The new cost row is

$$[\,0 \quad -10 \quad 0 \quad -20 \quad 0 \quad 0 \quad -9 \quad -11.5 \;|\; -11150\,]\,.$$

Thus, we pivot to optimality. The new production schedule is

$$(P, Y, B, T) = (6, 0, 16, 0).$$

3. If Fred and Martha are willing to pay double time for overtime work, how many hours of overtime should they purchase at this wage rate?

Initial Tableau:

	P	Y	B	T	S_1	S_2	S_3	S_4	b
acres	1	1	1	1	1	0	0	0	40
fertilizer	100	100	200	100	0	1	0	0	4000
machinetime	20	20	30	10	0	0	1	0	600
labor	30	20	20	20	0	0	0	1	500
	400	400	500	300	0	0	0	0	0

Optimal Tableaus:

P	Y	B	T	S_1	S_2	S_3	S_4	b
50	0	0	<u>100</u>	0	1	-10	5	500
2.5	1	0	2	0	0	$-.1$	.15	15
$-.5$	0	0	0	1	0	0	$-.05$	15
-1	0	1	-1	0	0	.1	$-.1$	10
-100	0	0	0	0	0	-10	-10	-11000
.5	0	0	1	0	.01	$-.1$	.05	5
1.5	1	0	0	0	$-.02$	.1	.05	5
$-.5$	0	0	0	1	0	0	$-.05$	15
$-.5$	0	1	0	0	.01	0	$-.05$	15
-100	0	0	0	0	0	-10	-10	-11000

Solution

The shadow price for labor hours is \$10 per hour. The current market value is \$5 per hour. Thus, one could pay up to \$15 per hour. Double–time is \$10 per hour, hence, they should contract for some over–time pay. The range analysis on labor hours follows:

$$\begin{array}{rcrcl} 500 & + & 5\theta & \geq & 0 \\ 15 & + & .15\theta & \geq & 0 \\ 15 & - & .05\theta & \geq & 0 \\ 10 & - & .1\theta & \geq & 0 \end{array} .$$

These inequalities combine to imply that $-100 \leq \theta \leq 100$. If we obtain more than 100 hours of over–time, then the column for the dual simplex pivot is the turnip column, hence, we also need to do the corresponding range analysis on the alternative optimal tableau, as follows:

$$\begin{array}{rcrcl} 5 & + & .05\theta & \geq & 0 \\ 5 & + & .05\theta & \geq & 0 \\ 15 & - & .05\theta & \geq & 0 \\ 15 & - & .05\theta & \geq & 0 \end{array} .$$

These inequalities combine to yield the range $-100 \leq \theta \leq 300$. Thus, we should buy at least 300 hours of labor at the double–time rate. A dual simplex pivot on either the third or the fourth row indicates that these extra hours will again yield multiple optimal solutions. But regardless of the new solution we obtain, the new shadow price for labor goes to zero and so we do not wish to contract for any more than the 300 hours of over–time.

4. Suppose Fred and Martha have the opportunity to purchase a small used tractor for \$1800 which has an operating cost of about \$10 an hour. If it is estimated that this tractor will contribute 150 hours of machine time, should Fred and Martha purchase it?

Initial Tableau:

	P	Y	B	T	S_1	S_2	S_3	S_4	b
acres	1	1	1	1	1	0	0	0	40
fertilizer	100	100	200	100	0	1	0	0	4000
machinetime	20	20	30	10	0	0	1	0	600
labor	30	20	20	20	0	0	0	1	500
	400	400	500	300	0	0	0	0	0

Optimal Tableaus:

P	Y	B	T	S_1	S_2	S_3	S_4	b
50	0	0	100	0	1	−10	5	500
2.5	1	0	2	0	0	−.1	.15	15
−.5	0	0	0	1	0	0	−.05	15
−1	0	1	−1	0	0	.1	−.1	10
−100	0	0	0	0	0	−10	−10	−11000
.5	0	0	1	0	.01	−.1	.05	5
1.5	1	0	0	0	−.02	.1	.05	5
−.5	0	0	0	1	0	0	−.05	15
−.5	0	1	0	0	.01	0	−.05	15
−100	0	0	0	0	0	−10	−10	−11000

Solution

First observe that if we simply replace 150 hours of our current machine time with those of the tractor, we get a savings of \$3000. For these 150 hours it will cost us \$1500 in operation costs. Thus, the tractor gives a true savings of \$1500 which is still less than the cost of the tractor. Thus, the only way for it to be efficient to purchase the tractor is if the added value of the extra machine hours makes it worthwhile. The shadow price for machine hours is \$10 per hour. Hence each extra hour is worth \$30 to us. We now do a range analysis to determine how many extra hours we can use:

$$\begin{array}{rcrcl} 500 & - & 10\theta & \geq & 0 \\ 15 & - & .1\theta & \geq & 0 \\ 15 & - & & \geq & 0 \\ 10 & + & .1\theta & \geq & 0 \end{array} .$$

These inequalities combine to imply that $-100 \leq \theta \leq 50$. The first row is the pivot row. This makes machine hours slack if we obtain more than 50 extra hours. Thus, if we buy the tractor the new profit will be

$$\text{profit} + \text{tractor savings} - \text{tractor costs} .$$

These values are

$$\begin{aligned} \text{profit} &= \$11500 \\ \text{tractor savings} &= \$3000 \\ \text{tractor costs} &= \$1500 \end{aligned}$$

yielding a net profit of \$ 13000. Since the change in profit is \$2000 we should buy the tractor.

5. The market for turnips has changed. They can now be sold for \$810 an acre. Under such circumstances what is the most that Fred and Martha are willing to pay for the tractor in question 4?

Initial Tableau:

	P	Y	B	T	S_1	S_2	S_3	S_4	b
acres	1	1	1	1	1	0	0	0	40
fertilizer	100	100	200	100	0	1	0	0	4000
machinetime	20	20	30	10	0	0	1	0	600
labor	30	20	20	20	0	0	0	1	500
	400	400	500	300	0	0	0	0	0

Optimal Tableaus:

P	Y	B	T	S_1	S_2	S_3	S_4	b
50	0	0	<u>100</u>	0	1	-10	5	500
2.5	1	0	2	0	0	$-.1$	.15	15
$-.5$	0	0	0	1	0	0	$-.05$	15
-1	0	1	-1	0	0	.1	$-.1$	10
-100	0	0	0	0	0	-10	-10	-11000
.5	0	0	1	0	.01	$-.1$	.05	5
1.5	1	0	0	0	$-.02$	.1	.05	5
$-.5$	0	0	0	1	0	0	$-.05$	15
$-.5$	0	1	0	0	.01	0	$-.05$	15
-100	0	0	0	0	0	-10	-10	-11000

Solution

We must first compute the change in the profitability of turnips:

$$\text{turnip production costs per acre}$$
$$= 100 + 10 + 200 + 100 = \$410\ .$$

Thus, the new turnip profit level is \$400 per acre. The best way to see the effect of this price change is to use the alternative optimal tableau of problem 1. This yields the new cost row

$$[\,-150 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0 \quad -15 \mid -11500\,]\ .$$

Again, alternative optima occur and we can choose to either produce or not produce yams. Regardless, the new shadow price for machine time is zero and the solution is not degenerate. Thus, we do not wish to purchase any more machine time. Consequently, we should pay no more than \$1500 for the tractor since this is the savings that occurs by using the tractor to replace co–op machine time.