Math 407A: Linear Optimization

Lecture 9
The Fundamental Theorem of Linear Programming
The Strong Duality Theorem
Complementary Slackness

Math Dept, University of Washington

The Two Phase Simplex Algorithm

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Every LP has the following three properties:

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(iii) *If it is bounded, then it has an optimal basic feasible solution.*

Duality Theory

$$\mathcal{P} \quad \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \le b,\ 0 \le x \end{array} \qquad\qquad \mathcal{D} \quad \begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \ge c,\ 0 \le y \end{array}$$

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What is the dual to the dual?

The Dual of the Dual

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c, \\ & 0 \leq y \end{array}$$

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$$\begin{array}{lll} \text{minimize} & b^T y & \text{Standard} \\ \text{subject to} & A^T y \geq c, & \Longrightarrow \\ & 0 \leq y & \text{form} \end{array}$$

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The dual of the dual is the primal.

The Weak Duality Theorem

Theorem:
If $x \in \mathbb{R}^n$ *is feasible for* $\mathcal{P}$ *and* $y \in \mathbb{R}^m$ *is feasible for* $\mathcal{D}$*, then*

$$c^T x \leq y^T A x \leq b^T y.$$

Thus, if $\mathcal{P}$ *is unbounded, then* $\mathcal{D}$ *is necessarily infeasible, and if* $\mathcal{D}$ *is unbounded, then* $\mathcal{P}$ *is necessarily infeasible. Moreover, if* $c^T \bar{x} = b^T \bar{y}$ *with* $\bar{x}$ *feasible for* $\mathcal{P}$ *and* $\bar{y}$ *feasible for* $\mathcal{D}$*, then* $\bar{x}$ *must solve* $\mathcal{P}$ *and* $\bar{y}$ *must solve* $\mathcal{D}$*.*

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If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

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Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

We combine the Weak Duality Theorem with the Fundamental Theorem of Linear Programming to obtain the *Strong Duality Theorem*.

The Strong Duality Theorem

Theorem:
If either $\mathcal{P}$ or $\mathcal{D}$ has a finite optimal value, then so does the other, the optimal values coincide, and optimal solutions to both $\mathcal{P}$ and $\mathcal{D}$ exist.

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$$\min f(x) = e^x$$

The optimal value is zero, but no solution exists.

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Proof:

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The optimal tableau is

$$\left[\begin{array}{cc|c} RA & R & Rb \\ \hline c^T - y^T A & -y^T & -y^T b \end{array}\right],$$

where we have already seen that y solves $\mathcal{D}$, and the optimal values coincide.

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This concludes the proof.

Complementary Slackness

Theorem: [WDT]
If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

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Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T\bar{x} = b^T\bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

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The SDT implies that x solves $\mathcal{P}$ and y solves $\mathcal{D}$ if and only if (x, y) is a $\mathcal{P}$-$\mathcal{D}$ feasible pair and

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We now examine the consequence of this equivalence.

Complementary Slackness

The equation $c^T x = y^T A x$ implies that

$$0 = x^T(A^T y - c) = \sum_{j=1}^{n} x_j(\sum_{i=1}^{m} a_{ij} y_i - c_j). \qquad (\clubsuit)$$

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$\mathcal{P}$-$\mathcal{D}$ feasibility gives

$$0 \le x_j \qquad \text{and} \qquad 0 \le \sum_{i=1}^{m} a_{ij} y_i - c_j \qquad \text{for } j = 1, \ldots, n.$$

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Hence, ($\clubsuit$) can only hold if

$$x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j) = 0 \quad \text{for} \quad j = 1, \ldots, n, \quad \text{or equivalently,}$$

$$x_j = 0 \qquad \text{or} \qquad \sum_{i=1}^{m} a_{ij}y_i = c_j \qquad \text{or both for } j = 1, \ldots, n.$$

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Similarly, the equation $y^T Ax = b^T y$ implies that

$$0 = y^T(b - Ax) = \sum_{i=1}^{m} y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j).$$

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Therefore, $\quad y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j) = 0 \quad i = 1, 2, \ldots, m.$

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Therefore, $\quad y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j) = 0 \quad i = 1, 2, \ldots, m.$

Hence,

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- $y_i = 0 \quad \text{or} \quad \sum_{j=1}^n a_{ij} x_j = b_i \quad \text{or both for } i = 1, \ldots, m.$

Complementary Slackness Theorem

Theorem:
The vector $x \in \mathbb{R}^n$ solves $\mathcal{P}$ and the vector $y \in \mathbb{R}^m$ solves $\mathcal{D}$ if and only if x is feasible for $\mathcal{P}$ and y is feasible for $\mathcal{D}$ and

(i) *either* $0 = x_j$ *or* $\sum_{i=1}^{m} a_{ij} y_i = c_j$ *or both for* $j = 1, \ldots, n$, *and*

(ii) *either* $0 = y_i$ *or* $\sum_{j=1}^{n} a_{ij} x_j = b_i$ *or both for* $i = 1, \ldots, m$.

Corollary to the Complementary Slackness Theorem

Corollary:
The vector $x \in \mathbb{R}^n$ solves $\mathcal{P}$ if and only if x is feasible for $\mathcal{P}$ and there exists a vector $y \in \mathbb{R}^m$ feasible for $\mathcal{D}$ and such that

(i) *if* $\sum_{j=1}^{n} a_{ij} x_j < b$, *then* $y_i = 0$, *for* $i = 1, \ldots, m$ *and*

(ii) *if* $0 < x_j$, *then* $\sum_{i=1}^{m} a_{ij} y_i = c_j$, *for* $j = 1, \ldots, n$.

Testing Optimality via Complementary Slackness

Does

$$x = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$$

solve the LP

$$\begin{array}{lrcrcrcrcrcl}
\text{maximize} & 7x_1 & + & 6x_2 & + & 5x_3 & - & 2x_4 & + & 3x_5 & & \\
\text{subject to} & x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \leq & 4 \\
& 4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \leq & 3 \\
& 2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \leq & 5 \\
& 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \leq & 1 \\
\end{array}$$
$$0 \leq x_1, x_2, x_3, x_4, x_5.$$

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& 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \leq & 1 & : & {\color{red}y_4} \\
& & & & & \multicolumn{5}{c}{0 \leq x_1, x_2, x_3, x_4, x_5.} & & & &
\end{array}$$

Testing Optimality via Complementary Slackness

The point

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must be feasible for the LP.

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must be feasible for the LP.
Plugging into the constraints we get

$$\begin{array}{rcrcrcrcrcl}
(0) & + & 3\left(\frac{4}{3}\right) & + & 5\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 2(0) & = & 4 \\
4(0) & + & 2\left(\frac{4}{3}\right) & - & 2\left(\frac{2}{3}\right) & + & \left(\frac{5}{3}\right) & + & (0) & = & 3 \\
2(0) & + & 4\left(\frac{4}{3}\right) & + & 4\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 5(0) & < & 5 \\
3(0) & + & \left(\frac{4}{3}\right) & + & 2\left(\frac{2}{3}\right) & - & \left(\frac{5}{3}\right) & - & 2(0) & = & 1.
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\end{array}$$

Can we use this information to construct a solution to the dual problem, (y_1, y_2, y_3, y_4)?

Testing Optimality via Complementary Slackness

Recall that

if $\sum_{j=1}^{n} a_{ij}x_j < b$, *then* $y_i = 0$, *for* $i = 1, \ldots, m$.

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We have just computed that

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Recall that

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$$\begin{array}{rcrcrcrcrcl}
(0) & + & 3\left(\frac{4}{3}\right) & + & 5\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 2(0) & = & 4 & : & y_1 \\
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3(0) & + & \left(\frac{4}{3}\right) & + & 2\left(\frac{2}{3}\right) & - & \left(\frac{5}{3}\right) & - & 2(0) & = & 1 & : \ {\color{red}y_4}
\end{array}$$

So $y_3 = 0$.

Testing Optimality via Complementary Slackness

Also recall that

if $0 < x_j$, *then* $\sum_{i=1}^{m} a_{ij} y_i = c_j$, *for* $j = 1, \ldots, n$.

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Testing Optimality via Complementary Slackness

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Hence,

$$3y_1 + 2y_2 + 4y_3 + y_4 = 6 \qquad (x_2 = \tfrac{4}{3} > 0)$$

Testing Optimality via Complementary Slackness

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Hence,

$$\begin{array}{ccccccccccc} 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & = & 6 & & (x_2 = \frac{4}{3} > 0) \\ 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & = & 5 & & (x_3 = \frac{2}{3} > 0) \end{array}$$

Testing Optimality via Complementary Slackness

Also recall that

if $0 < x_j$, *then* $\sum_{i=1}^{m} a_{ij} y_i = c_j$, *for* $j = 1, \ldots, n$.

Hence,

$$\begin{array}{rcrcrcrcrl}
3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & = & 6 & \qquad (x_2 = \frac{4}{3} > 0) \\
5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & = & 5 & \qquad (x_3 = \frac{2}{3} > 0) \\
-2y_1 & + & y_2 & - & 2y_3 & - & y_4 & = & -2 & \qquad (x_4 = \frac{5}{3} > 0)
\end{array}$$

Testing Optimality via Complementary Slackness

Combining these observations gives the system

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 5 & -2 & 4 & 2 \\ -2 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -2 \\ 0 \end{pmatrix},$$

which any dual solution must satisfy.

Testing Optimality via Complementary Slackness

Combining these observations gives the system

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which any dual solution must satisfy.
This is a square system that we can try to solve for y.

Testing Optimality via Complementary Slackness

3	2	4	1	6	
5	−2	4	2	5	
−2	1	−2	−1	−2	
0	0	1	0	0	
3	2	0	1	6	$r_1 - 4r_4$
5	−2	0	2	5	$r_2 - 4r_4$
−2	1	0	−1	−2	$r_3 + 2r_4$
0	0	1	0	0	
1	3	0	0	4	$r_1 + r_3$
1	0	0	0	1	$r_2 + 2r_3$
−2	1	0	−1	−2	
0	0	1	0	0	

Testing Optimality via Complementary Slackness

3	2	4	1	6	
5	-2	4	2	5	
-2	1	-2	-1	-2	
0	0	1	0	0	
3	2	0	1	6	$r_1 - 4r_4$
5	-2	0	2	5	$r_2 - 4r_4$
-2	1	0	-1	-2	$r_3 + 2r_4$
0	0	1	0	0	
1	3	0	0	4	$r_1 + r_3$
1	0	0	0	1	$r_2 + 2r_3$
-2	1	0	-1	-2	
0	0	1	0	0	

1	3	0	0	4	$r_1 + r_3$
1	0	0	0	1	$r_2 + 2r_3$
-2	1	0	-1	-2	
0	0	1	0	0	
0	3	0	0	3	$r_1 - r_2$
1	0	0	0	1	
0	1	0	-1	0	$r_3 + 2r_2$
0	0	1	0	0	
1	0	0	0	1	r_2
0	1	0	0	1	$\frac{1}{3}r_1$
0	0	1	0	0	r_4
0	0	0	1	1	$-r_3 + \frac{1}{3}r_1$

Testing Optimality via Complementary Slackness

$$
\begin{array}{rrrr|r l}
\hline
3 & 2 & 4 & 1 & 6 & \\
5 & -2 & 4 & 2 & 5 & \\
-2 & 1 & -2 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
3 & 2 & 0 & 1 & 6 & r_1 - 4r_4 \\
5 & -2 & 0 & 2 & 5 & r_2 - 4r_4 \\
-2 & 1 & 0 & -1 & -2 & r_3 + 2r_4 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 3 & 0 & 0 & 4 & r_1 + r_3 \\
1 & 0 & 0 & 0 & 1 & r_2 + 2r_3 \\
-2 & 1 & 0 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
\end{array}
\qquad
\begin{array}{rrrr|r l}
\hline
1 & 3 & 0 & 0 & 4 & r_1 + r_3 \\
1 & 0 & 0 & 0 & 1 & r_2 + 2r_3 \\
-2 & 1 & 0 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
0 & 3 & 0 & 0 & 3 & r_1 - r_2 \\
1 & 0 & 0 & 0 & 1 & \\
0 & 1 & 0 & -1 & 0 & r_3 + 2r_2 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & 1 & r_2 \\
0 & 1 & 0 & 0 & 1 & \frac{1}{3}r_1 \\
0 & 0 & 1 & 0 & 0 & r_4 \\
0 & 0 & 0 & 1 & 1 & -r_3 + \frac{1}{3}r_1 \\
\hline
\end{array}
$$

This gives the solution $(y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$.

Testing Optimality via Complementary Slackness

$$
\begin{array}{rrrr|r}
3 & 2 & 4 & 1 & 6 \\
5 & -2 & 4 & 2 & 5 \\
-2 & 1 & -2 & -1 & -2 \\
0 & 0 & 1 & 0 & 0 \\ \hline
3 & 2 & 0 & 1 & 6 \\
5 & -2 & 0 & 2 & 5 \\
-2 & 1 & 0 & -1 & -2 \\
0 & 0 & 1 & 0 & 0 \\ \hline
1 & 3 & 0 & 0 & 4 \\
1 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -1 & -2 \\
0 & 0 & 1 & 0 & 0
\end{array}
\quad
\begin{array}{l}
\\ \\ \\ \\
r_1 - 4r_4 \\
r_2 - 4r_4 \\
r_3 + 2r_4 \\
\\
r_1 + r_3 \\
r_2 + 2r_3 \\
\\
\\
\end{array}
$$

$$
\begin{array}{rrrr|r}
1 & 3 & 0 & 0 & 4 \\
1 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -1 & -2 \\
0 & 0 & 1 & 0 & 0 \\ \hline
0 & 3 & 0 & 0 & 3 \\
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{array}
\quad
\begin{array}{l}
r_1 + r_3 \\
r_2 + 2r_3 \\
\\
\\
r_1 - r_2 \\
\\
r_3 + 2r_2 \\
\\
r_2 \\
\frac{1}{3}r_1 \\
r_4 \\
-r_3 + \frac{1}{3}r_1
\end{array}
$$

This gives the solution $(y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$.

Is this dual feasible?

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & \multicolumn{5}{c}{0 \leq y_1, y_2, y_3, y_4.} & &
\end{array}
$$

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & \multicolumn{5}{c}{0 \leq y_1, y_2, y_3, y_4.} & &
\end{array}$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & \multicolumn{5}{c}{0 \leq y_1, y_2, y_3, y_4.} & &
\end{array}
$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & 0 \leq y_1, y_2, y_3, y_4. & & & & & &
\end{array}$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

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First: $1 + 4 + 0 + 3 = 8 > 7$

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & \multicolumn{5}{c}{0 \leq y_1, y_2, y_3, y_4.} & &
\end{array}$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

First: $1 + 4 + 0 + 3 = 8 > 7$

Fifth: $2 + 1 + 0 - 2 = 1 \not\geq 3$, the fifth dual inequality is violated.

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & \multicolumn{5}{c}{0 \leq y_1, y_2, y_3, y_4.} & &
\end{array}
$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

First: $1 + 4 + 0 + 3 = 8 > 7$

Fifth: $2 + 1 + 0 - 2 = 1 \not\geq 3$, the fifth dual inequality is violated.

Hence, $x = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$ cannot be optimal!

Example: Testing Optimality via Complementary Slackness

Does the point $x = (1,1,1,0)$ solve the following LP?

$$
\begin{array}{lrrrrcl}
\text{maximize} & 4x_1 & +2x_2 & +2x_3 & +4x_4 & & \\
\text{subject to} & x_1 & +3x_2 & +2x_3 & +\ x_4 & \leq & 7 \\
 & x_1 & +\ x_2 & +\ x_3 & +2x_4 & \leq & 3 \\
 & 2x_1 & & +\ x_3 & +\ x_4 & \leq & 3 \\
 & x_1 & +\ x_2 & & +2x_4 & \leq & 2 \\
 & 0 \leq & x_1, x_2, & x_3, x_4 & & &
\end{array}
$$