Math 407A: Linear Optimization

Lecture 9
The Fundamental Theorem of Linear Programming
The Strong Duality Theorem
Complementary Slackness

Math Dept, University of Washington
The Two Phase Simplex Algorithm

The Fundamental Theorem of Linear Programming

Duality Theory Revisited

Complementary Slackness
The Two Phase Simplex Algorithm

Phase I  Formulate and solve the auxiliary problem. Two outcomes are possible:

(i) The optimal value in the auxiliary problem is positive. In this case the original problem is infeasible.
(ii) The optimal value is zero and an initial feasible tableau for the original problem is obtained.

Phase II  If the original problem is feasible, apply the simplex algorithm to the initial feasible tableau obtained from Phase I above. Again, two outcomes are possible:

(i) The LP is determined to be unbounded.
(ii) An optimal basic feasible solution is obtained.
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The Fundamental Theorem of linear Programming

Theorem:

*Every LP has the following three properties:*
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The Fundamental Theorem of linear Programming

**Theorem:**

*Every LP has the following three properties:*

(i) *If it has no optimal solution, then it is either infeasible or unbounded.*

(ii) *If it has a feasible solution, then it has a basic feasible solution.*

(iii) *If it is bounded, then it has an optimal basic feasible solution.*
Duality Theory

\[ \mathcal{P} \quad \text{maximize} \quad c^T x \quad \text{subject to} \quad Ax \leq b, \ 0 \leq x \]

\[ \mathcal{D} \quad \text{minimize} \quad b^T y \quad \text{subject to} \quad A^T y \geq c, \ 0 \leq y \]
Duality Theory

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subject to \[ Ax \leq b, \ 0 \leq x \]

\[ \mathcal{D} \quad \text{minimize} \quad b^T y \]
subject to \[ A^T y \geq c, \ 0 \leq y \]

What is the dual to the dual?
The Dual of the Dual

minimize \( b^T y \)
subject to \( A^T y \geq c \),
\( 0 \leq y \)

\[ \text{Standard} \Rightarrow \text{form} \]

\[ \begin{array}{ll}
\min & -b^T y \\
\text{subject to} & -(A^T) y \leq -(c) \\
& 0 \leq y \\
\end{array} \]

\[ \begin{array}{ll}
\min & -(c)^T x \\
\text{subject to} & -(A^T) x \geq -(b) \\
& 0 \leq x \\
\end{array} \]
The Dual of the Dual

\[
\begin{align*}
\text{minimize} & \quad b^T y & \text{Standard} \\
\text{subject to} & \quad A^T y \geq c, & \\
& \quad 0 \leq y & \quad \Rightarrow \\
& \quad 0 \leq y & \quad \text{form}
\end{align*}
\]
The Dual of the Dual

\[
\begin{align*}
\text{minimize} & \quad b^T y \\
\text{subject to} & \quad A^T y \geq c, \\
& \quad 0 \leq y
\end{align*}
\]

Standard form

\[
\begin{align*}
\text{− maximize} & \quad (−b)^T y \\
\text{subject to} & \quad (−A^T)y \leq (−c), \\
& \quad 0 \leq y.
\end{align*}
\]
The Dual of the Dual

minimize \( b^T y \)
subject to \( A^T y \geq c, \)
\( 0 \leq y \)

Standard form

\[ \Rightarrow \]

\( \] – minimize \( (-b)^T y \)
subject to \( (-A^T)y \leq (-c), \)
\( 0 \leq y. \)

minimize \( (-c)^T x \)
subject to \( (-A^T)^T x \geq (-b), \)
\( 0 \leq x \)

The dual of the dual is the primal.
The Dual of the Dual

\begin{align*}
\text{minimize} \quad & b^T y \\
\text{subject to} \quad & A^T y \geq c, \\
\quad & 0 \leq y
\end{align*}

Standard form

\begin{align*}
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\begin{align*}
\text{maximize} \quad & c^T x \\
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\[
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\]

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\]

The dual of the dual is the primal.
The Weak Duality Theorem

**Theorem:**
If $x \in \mathbb{R}^n$ is feasible for $P$ and $y \in \mathbb{R}^m$ is feasible for $D$, then

$$c^T x \leq y^T A x \leq b^T y.$$ 

Thus, if $P$ is unbounded, then $D$ is necessarily infeasible, and if $D$ is unbounded, then $P$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $P$ and $\bar{y}$ feasible for $D$, then $\bar{x}$ must solve $P$ and $\bar{y}$ must solve $D$. 

We combine the Weak Duality Theorem with the Fundamental Theorem of Linear Programming to obtain the Strong Duality Theorem.
**The Weak Duality Theorem**

**Theorem:**

If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \leq y^T A x \leq b^T y.$$ 

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

We combine the Weak Duality Theorem with the Fundamental Theorem of Linear Programming to obtain the *Strong Duality Theorem*. 
The Strong Duality Theorem

**Theorem:**
If either $P$ or $D$ has a finite optimal value, then so does the other, the optimal values coincide, and optimal solutions to both $P$ and $D$ exist.
The Strong Duality Theorem

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**Remark:** In general a finite optimal value does not imply the existence of a solution.
The Strong Duality Theorem

**Theorem:**
If either \( P \) or \( D \) has a finite optimal value, then so does the other, the optimal values coincide, and optimal solutions to both \( P \) and \( D \) exist.

**Remark:** In general a finite optimal value does not imply the existence of a solution.

\[
\min f(x) = e^x
\]

The optimal value is zero, but no solution exists.
The Strong Duality Theorem

Proof:
Since the dual of the dual is the primal, we may as well assume that the primal has a finite optimal value.
The Strong Duality Theorem

**Proof:**
Since the dual of the dual is the primal, we may as well assume that the primal has a finite optimal value.

The Fundamental Theorem of Linear Programming says that an optimal basic feasible solution exists.
The Strong Duality Theorem

**Proof:**
Since the dual of the dual is the primal, we may as well assume that the primal has a finite optimal value.

The Fundamental Theorem of Linear Programming says that an optimal basic feasible solution exists.

The optimal tableau is

\[
\begin{bmatrix}
    RA & R & Rb \\
    c^T - y^T A & -y^T & -y^T b
\end{bmatrix},
\]

where we have already seen that \( y \) solves \( \mathcal{D} \), and the optimal values coincide.
The Strong Duality Theorem

**Proof:**
Since the dual of the dual is the primal, we may as well assume that the primal has a finite optimal value.

The Fundamental Theorem of Linear Programming says that an optimal basic feasible solution exists.

The optimal tableau is

$$\begin{bmatrix}
    RA & R & Rb \\
    c^T - y^T A & -y^T & -y^T b
\end{bmatrix},$$

where we have already seen that $y$ solves $\mathcal{D}$, and the optimal values coincide.

This concludes the proof.
Complementary Slackness

**Theorem: [WDT]**

If \( x \in \mathbb{R}^n \) is feasible for \( P \) and \( y \in \mathbb{R}^m \) is feasible for \( D \), then

\[
    c^T x \leq y^T Ax \leq b^T y.
\]

Thus, if \( P \) is unbounded, then \( D \) is necessarily infeasible, and if \( D \) is unbounded, then \( P \) is necessarily infeasible. Moreover, if \( c^T \bar{x} = b^T \bar{y} \) with \( \bar{x} \) feasible for \( P \) and \( \bar{y} \) feasible for \( D \), then \( \bar{x} \) must solve \( P \) and \( \bar{y} \) must solve \( D \).
Complementary Slackness

**Theorem: [WDT]**

If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

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Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

The SDT implies that $x$ solves $\mathcal{P}$ and $y$ solves $\mathcal{D}$ if and only if $(x, y)$ is a $\mathcal{P}$-$\mathcal{D}$ feasible pair and

$$c^T x = y^T A x = b^T y.$$
Complementary Slackness

**Theorem:** [WDT]  
If \( x \in \mathbb{R}^n \) is feasible for \( \mathcal{P} \) and \( y \in \mathbb{R}^m \) is feasible for \( \mathcal{D} \), then 

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c^T x \leq y^T Ax \leq b^T y.
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Thus, if \( \mathcal{P} \) is unbounded, then \( \mathcal{D} \) is necessarily infeasible, and if \( \mathcal{D} \) is unbounded, then \( \mathcal{P} \) is necessarily infeasible. Moreover, if \( c^T \bar{x} = b^T \bar{y} \) with \( \bar{x} \) feasible for \( \mathcal{P} \) and \( \bar{y} \) feasible for \( \mathcal{D} \), then \( \bar{x} \) must solve \( \mathcal{P} \) and \( \bar{y} \) must solve \( \mathcal{D} \).

The SDT implies that \( x \) solves \( \mathcal{P} \) and \( y \) solves \( \mathcal{D} \) if and only if \((x, y)\) is a \( \mathcal{P}-\mathcal{D} \) feasible pair and

\[
c^T x = y^T Ax = b^T y.
\]

We now examine the consequence of this equivalence.
Complementary Slackness

The equation $c^T x = y^T A x$ implies that

$$0 = x^T (A^T y - c) = \sum_{j=1}^{n} x_j \left( \sum_{i=1}^{m} a_{ij} y_i - c_j \right).$$
Complementary Slackness

The equation \( c^T x = y^T Ax \) implies that

\[
0 = x^T (A^T y - c) = \sum_{j=1}^{n} x_j (\sum_{i=1}^{m} a_{ij} y_i - c_j).
\] (♣)

\( \mathcal{P} \)-\( \mathcal{D} \) feasibility gives

\[
0 \leq x_j \quad \text{and} \quad 0 \leq \sum_{i=1}^{m} a_{ij} y_i - c_j \quad \text{for } j = 1, \ldots, n.
\]
Complementary Slackness

The equation \( c^T x = y^T A x \) implies that

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\]

\( P-D \) feasibility gives

\[
0 \leq x_j \quad \text{and} \quad 0 \leq \sum_{i=1}^{m} a_{ij} y_i - c_j \quad \text{for } j = 1, \ldots, n.
\]

Hence, (♣) can only hold if

\[
x_j \left( \sum_{i=1}^{m} a_{ij} y_i - c_j \right) = 0 \quad \text{for } j = 1, \ldots, n, \quad \text{or equivalently,}
\]

\[
x_j = 0 \quad \text{or} \quad \sum_{i=1}^{m} a_{ij} y_i = c_j \quad \text{or both for } j = 1, \ldots, n.
\]
Complementary Slackness

Similarly, the equation \( y^T A x = b^T y \) implies that

\[
0 = y^T (b - Ax) = \sum_{i=1}^{m} y_i (b_i - \sum_{j=1}^{n} a_{ij} x_j).
\]
Complementary Slackness

Similarly, the equation $y^T Ax = b^T y$ implies that

$$0 = y^T (b - Ax) = \sum_{i=1}^{m} y_i (b_i - \sum_{j=1}^{n} a_{ij} x_j).$$

$$\left(\begin{array}{c} 0 \leq y_i \\ 0 \leq b_i - \sum_{j=1}^{n} a_{ij} x_j \end{array}\right)$$
Complementary Slackness

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$$0 = y^T (b - Ax) = \sum_{i=1}^{m} y_i (b_i - \sum_{j=1}^{n} a_{ij} x_j).$$

Therefore, $y_i (b_i - \sum_{j=1}^{n} a_{ij} x_j) = 0$ for $i = 1, 2, \ldots, m$. 
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$$0 = y^T (b - Ax) = \sum_{i=1}^{m} y_i (b_i - \sum_{j=1}^{n} a_{ij} x_j).$$

Therefore, $y_i (b_i - \sum_{j=1}^{n} a_{ij} x_j) = 0 \quad i = 1, 2, \ldots, m.$

Hence,

$$y_i = 0 \quad \text{or} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{or both for } i = 1, \ldots, m.$$
Complementary Slackness

\[ c^T x = y^T A x = b^T y \]

\[ \iff \]

\[
\begin{align*}
\text{for } j &= 1, \ldots, n, \\
\text{or both for } j &= 1, \ldots, n.
\end{align*}
\]

\[
\begin{align*}
\text{for } i &= 1, \ldots, m, \\
\text{or both for } i &= 1, \ldots, m.
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Complementary Slackness

\[ c^T x = y^T Ax = b^T y \]

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\[ \text{\textbullet } x_j = 0 \quad \text{or} \quad \sum_{i=1}^{m} a_{ij} y_i = c_j \quad \text{or both for } j = 1, \ldots, n. \]
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\[ y_i = 0 \quad \text{or} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{or both for } i = 1, \ldots, m. \]
Complementary Slackness Theorem

Theorem:
The vector \( x \in \mathbb{R}^n \) solves \( \mathcal{P} \) and the vector \( y \in \mathbb{R}^m \) solves \( \mathcal{D} \) if and only if \( x \) is feasible for \( \mathcal{P} \) and \( y \) is feasible for \( \mathcal{D} \) and

(i) either \( 0 = x_j \) or \( \sum_{i=1}^{m} a_{ij}y_i = c_j \) or both for \( j = 1, \ldots, n \), and

(ii) either \( 0 = y_i \) or \( \sum_{j=1}^{n} a_{ij}x_j = b_i \) or both for \( i = 1, \ldots, m \).
Corollary to the Complementary Slackness Theorem

**Corollary:**

The vector \( x \in \mathbb{R}^n \) solves \( \mathcal{P} \) if and only if \( x \) is feasible for \( \mathcal{P} \) and there exists a vector \( y \in \mathbb{R}^m \) feasible for \( \mathcal{D} \) and such that

(i) if \( \sum_{j=1}^{n} a_{ij} x_j < b \), then \( y_i = 0 \), for \( i = 1, \ldots, m \) and

(ii) if \( 0 < x_j \), then \( \sum_{i=1}^{m} a_{ij} y_i = c_j \), for \( j = 1, \ldots, n \).
Testing Optimality via Complementary Slackness

Does

\[ x = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0) \]

solve the LP

maximize \[ 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5 \]

subject to \[ x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 \leq 4 \]
\[ 4x_1 + 2x_2 - 2x_3 + x_4 + x_5 \leq 3 \]
\[ 2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 \leq 5 \]
\[ 3x_1 + x_2 + 2x_3 - x_4 - 2x_5 \leq 1 \]
\[ 0 \leq x_1, x_2, x_3, x_4, x_5. \]
Testing Optimality via Complementary Slackness

Does

\[ x = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0) \]

solve the LP

maximize \[ 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5 \]

subject to \[ x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 \leq 4 : y_1 \]
\[ 4x_1 + 2x_2 - 2x_3 + x_4 + x_5 \leq 3 : y_2 \]
\[ 2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 \leq 5 : y_3 \]
\[ 3x_1 + x_2 + 2x_3 - x_4 - 2x_5 \leq 1 : y_4 \]
\[ 0 \leq x_1, x_2, x_3, x_4, x_5. \]
Testing Optimality via Complementary Slackness

The point

\[ x = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0) \]

must be feasible for the LP.
Testing Optimality via Complementary Slackness

The point

\[ x = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0) \]

must be feasible for the LP.

Plugging into the constraints we get

\[
\begin{align*}
0 + 3 \left( \frac{4}{3} \right) + 5 \left( \frac{2}{3} \right) - 2 \left( \frac{5}{3} \right) + 2(0) &= 4 \\
4(0) + 2 \left( \frac{4}{3} \right) - 2 \left( \frac{2}{3} \right) + \left( \frac{5}{3} \right) + (0) &= 3 \\
2(0) + 4 \left( \frac{4}{3} \right) + 4 \left( \frac{2}{3} \right) - 2 \left( \frac{5}{3} \right) + 5(0) &< 5 \\
3(0) + \left( \frac{4}{3} \right) + 2 \left( \frac{2}{3} \right) - \left( \frac{5}{3} \right) - 2(0) &= 1.
\end{align*}
\]
Testing Optimality via Complementary Slackness

The point

\[ x = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0) \]

must be feasible for the LP.

Plugging into the constraints we get

\[
\begin{align*}
(0) + 3 \left( \frac{4}{3} \right) + 5 \left( \frac{2}{3} \right) - 2 \left( \frac{5}{3} \right) + 2(0) &= 4 \\
4(0) + 2 \left( \frac{4}{3} \right) - 2 \left( \frac{2}{3} \right) + \left( \frac{5}{3} \right) + (0) &= 3 \\
2(0) + 4 \left( \frac{4}{3} \right) + 4 \left( \frac{2}{3} \right) - 2 \left( \frac{5}{3} \right) + 5(0) &< 5 \\
3(0) + \left( \frac{4}{3} \right) + 2 \left( \frac{2}{3} \right) - \left( \frac{5}{3} \right) - 2(0) &= 1.
\end{align*}
\]

Can we use this information to construct a solution to the dual problem, \((y_1, y_2, y_3, y_4)\)?
Recall that
\[
\text{if } \sum_{j=1}^{n} a_{ij} x_j < b, \text{ then } y_i = 0, \text{ for } i = 1, \ldots, m.
\]
Recall that
\[ \text{if } \sum_{j=1}^{n} a_{ij}x_j < b, \text{ then } y_i = 0, \text{ for } i = 1, \ldots, m. \]

We have just computed that
\[
\begin{align*}
(0) + 3\left(\frac{4}{3}\right) + 5\left(\frac{2}{3}\right) - 2\left(\frac{5}{3}\right) + 2(0) &= 4 \\
4(0) + 2\left(\frac{4}{3}\right) - 2\left(\frac{2}{3}\right) + \left(\frac{5}{3}\right) + (0) &= 3 \\
2(0) + 4\left(\frac{4}{3}\right) + 4\left(\frac{2}{3}\right) - 2\left(\frac{5}{3}\right) + 5(0) &< 5 \\
3(0) + \left(\frac{4}{3}\right) + 2\left(\frac{2}{3}\right) - \left(\frac{5}{3}\right) - 2(0) &= 1
\end{align*}
\]
Recall that if $\sum_{j=1}^{n} a_{ij}x_j < b$, then $y_i = 0$, for $i = 1, \ldots, m$.

We have just computed that

$$
egin{align*}
(0) & + 3 \left(\frac{4}{3}\right) + 5 \left(\frac{2}{3}\right) - 2 \left(\frac{5}{3}\right) + 2(0) = 4 : y_1 \\
4(0) & + 2 \left(\frac{4}{3}\right) - 2 \left(\frac{2}{3}\right) + \left(\frac{5}{3}\right) + (0) = 3 : y_2 \\
2(0) & + 4 \left(\frac{4}{3}\right) + 4 \left(\frac{2}{3}\right) - 2 \left(\frac{5}{3}\right) + 5(0) < 5 : y_3 \\
3(0) & + \left(\frac{4}{3}\right) + 2 \left(\frac{2}{3}\right) - \left(\frac{5}{3}\right) - 2(0) = 1 : y_4
\end{align*}
$$
Recall that if $\sum_{j=1}^n a_{ij}x_j < b$, then $y_i = 0$, for $i = 1, \ldots, m$.

We have just computed that

$$
(0) + 3\left(\frac{4}{3}\right) + 5\left(\frac{2}{3}\right) - 2\left(\frac{5}{3}\right) + 2(0) = 4 : y_1
$$

$$
4(0) + 2\left(\frac{4}{3}\right) - 2\left(\frac{2}{3}\right) + \left(\frac{5}{3}\right) + (0) = 3 : y_2
$$

$$
2(0) + 4\left(\frac{4}{3}\right) + 4\left(\frac{2}{3}\right) - 2\left(\frac{5}{3}\right) + 5(0) < 5 : y_3
$$

$$
3(0) + \left(\frac{4}{3}\right) + 2\left(\frac{2}{3}\right) - \left(\frac{5}{3}\right) - 2(0) = 1 : y_4
$$

So $y_3 = 0$. 
Testing Optimality via Complementary Slackness

Also recall that

\[ \text{if } 0 < x_j, \text{ then } \sum_{i=1}^{m} a_{ij} y_i = c_j, \text{ for } j = 1, \ldots, n. \]
Testing Optimality via Complementary Slackness

Also recall that

if $0 < x_j$, then $\sum_{i=1}^{m} a_{ij} y_i = c_j$, for $j = 1, \ldots, n$.

Hence,
Also recall that

\[ \text{if } 0 < x_j, \text{ then } \sum_{i=1}^{m} a_{ij} y_i = c_j, \text{ for } j = 1, \ldots, n. \]

Hence,

\[ 3y_1 + 2y_2 + 4y_3 + y_4 = 6 \quad (x_2 = \frac{4}{3} > 0) \]
Testing Optimality via Complementary Slackness

Also recall that

\[ \text{if } 0 < x_j, \text{ then } \sum_{i=1}^{m} a_{ij}y_i = c_j, \text{ for } j = 1, \ldots, n. \]

Hence,

\[ 3y_1 + 2y_2 + 4y_3 + y_4 = 6 \quad (x_2 = \frac{4}{3} > 0) \]

\[ 5y_1 - 2y_2 + 4y_3 + 2y_4 = 5 \quad (x_3 = \frac{2}{3} > 0) \]
Testing Optimality via Complementary Slackness

Also recall that

\[ \text{if } 0 < x_j, \text{ then } \sum_{i=1}^{m} a_{ij}y_i = c_j, \text{ for } j = 1, \ldots, n. \]

Hence,

\[
\begin{align*}
3y_1 & + 2y_2 & + 4y_3 & + y_4 & = 6 & (x_2 = \frac{4}{3} > 0) \\
5y_1 & - 2y_2 & + 4y_3 & + 2y_4 & = 5 & (x_3 = \frac{2}{3} > 0) \\
-2y_1 & + y_2 & - 2y_3 & - y_4 & = -2 & (x_4 = \frac{5}{3} > 0)
\end{align*}
\]
Combining these observations gives the system

\[
\begin{bmatrix}
3 & 2 & 4 & 1 \\
5 & -2 & 4 & 2 \\
-2 & 1 & -2 & -1 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\end{pmatrix}
= 
\begin{pmatrix}
6 \\
5 \\
-2 \\
0 \\
\end{pmatrix},
\]

which any dual solution must satisfy.
Combining these observations gives the system

\[
\begin{bmatrix}
3 & 2 & 4 & 1 \\
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\end{pmatrix}
= 
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-2 \\
0 \\
\end{pmatrix},
\]

which any dual solution must satisfy. This is a square system that we can try to solve for \( y \).
Testing Optimality via Complementary Slackness

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This gives the solution \((y_1, y_2, y_3, y_4) = (1, 1, 0, 1)\).

Is this dual feasible?
Testing Optimality via Complementary Slackness

\[
\begin{array}{cccc}
3 & 2 & 4 & 1 \\
5 & -2 & 4 & 2 \\
-2 & 1 & -2 & -1 \\
0 & 0 & 1 & 0 \\
\hline
3 & 2 & 0 & 1 \\
5 & -2 & 0 & 2 \\
-2 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-2 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\hline
0 & 3 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

This gives the solution \((y_1, y_2, y_3, y_4) = (1, 1, 0, 1)\). Is this dual feasible?
Testing Optimality via Complementary Slackness

This gives the solution \( (y_1, y_2, y_3, y_4) = (1, 1, 0, 1) \).
Testing Optimality via Complementary Slackness

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This gives the solution \((y_1, y_2, y_3, y_4) = (1, 1, 0, 1)\).

Is this dual feasible?

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\(r_1 + r_3\)
\(r_2 + 2r_3\)
\(r_1 - r_2\)
\(r_3 + 2r_2\)
\(r_2\)
\(\frac{1}{3} r_1\)
\(r_4\)
\(- r_3 + \frac{1}{3} r_1\)
Testing Optimality via Complementary Slackness

\[ y = (y_1, y_2, y_3, y_4) = (1, 1, 0, 1) \]

\[
\begin{align*}
\text{minimize} & \quad 4y_1 + 3y_2 + 5y_3 + y_4 \\
\text{subject to} & \quad y_1 + 4y_2 + 2y_3 + 3y_4 \geq 7 \\
& \quad 3y_1 + 2y_2 + 4y_3 + y_4 \geq 6 \\
& \quad 5y_1 - 2y_2 + 4y_3 + 2y_4 \geq 5 \\
& \quad -2y_1 + y_2 - 2y_3 - y_4 \geq -2 \\
& \quad 2y_1 + y_2 + 5y_3 - 2y_4 \geq 3 \\
& \quad 0 \leq y_1, y_2, y_3, y_4.
\end{align*}
\]

Clearly, \(0 \leq y\) and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality. We need to check the first and fifth inequalities.

First: \(1 + 4 + 0 + 3 = 8 > 7\)

Fifth: \(2 + 1 + 0 - 2 = 1 \not\geq 3\), the fifth dual inequality is violated.

Hence, \(x = (0, 1, 2, 3, 0)\) cannot be optimal!
Testing Optimality via Complementary Slackness

\[ y = (y_1, y_2, y_3, y_4) = (1, 1, 0, 1) \]

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Testing Optimality via Complementary Slackness

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Clearly, \(0 \leq y\) and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.
Testing Optimality via Complementary Slackness

\[ y = (y_1, y_2, y_3, y_4) = (1, 1, 0, 1) \]

minimize \[ 4y_1 + 3y_2 + 5y_3 + y_4 \]
subject to \[ \begin{align*}
y_1 + 4y_2 + 2y_3 + 3y_4 & \geq 7 \\
3y_1 + 2y_2 + 4y_3 + y_4 & \geq 6 \\
5y_1 - 2y_2 + 4y_3 + 2y_4 & \geq 5 \\
-2y_1 + y_2 - 2y_3 - y_4 & \geq -2 \\
2y_1 + y_2 + 5y_3 - 2y_4 & \geq 3 \\
0 & \leq y_1, y_2, y_3, y_4.
\end{align*} \]

Clearly, \( 0 \leq y \) and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.
First: \( 1 + 4 + 0 + 3 = 8 > 7 \)
Testing Optimality via Complementary Slackness

\[ y = (y_1, y_2, y_3, y_4) = (1, 1, 0, 1) \]

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\begin{align*}
\text{minimize} & \quad 4y_1 + 3y_2 + 5y_3 + y_4 \\
\text{subject to} & \quad y_1 + 4y_2 + 2y_3 + 3y_4 \geq 7 \\
& \quad 3y_1 + 2y_2 + 4y_3 + y_4 \geq 6 \\
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\end{align*}
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Clearly, \( 0 \leq y \) and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

First: \( 1 + 4 + 0 + 3 = 8 > 7 \)

Fifth: \( 2 + 1 + 0 - 2 = 1 \not\geq 3 \), the fifth dual inequality is violated.
Testing Optimality via Complementary Slackness

\[ y = (y_1, y_2, y_3, y_4) = (1, 1, 0, 1) \]

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\begin{align*}
\text{minimize} & \quad 4y_1 + 3y_2 + 5y_3 + y_4 \\
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& \quad 5y_1 - 2y_2 + 4y_3 + 2y_4 \geq 5 \\
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Clearly, \(0 \leq y\) and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

First: \(1 + 4 + 0 + 3 = 8 > 7\)

Fifth: \(2 + 1 + 0 - 2 = 1 \ngeq 3\), the fifth dual inequality is violated.

Hence, \(x = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)\) cannot be optimal!
Example: Testing Optimality via Complementary Slackness

Does the point $x = (1, 1, 1, 0)$ solve the following LP?

\[
\begin{align*}
\text{maximize} & \quad 4x_1 + 2x_2 + 2x_3 + 4x_4 \\
\text{subject to} & \quad x_1 + 3x_2 + 2x_3 + x_4 \leq 7 \\
& \quad x_1 + x_2 + x_3 + 2x_4 \leq 3 \\
& \quad 2x_1 + x_3 + x_4 \leq 3 \\
& \quad x_1 + x_2 + 2x_4 \leq 2 \\
& \quad 0 \leq x_1, x_2, x_3, x_4
\end{align*}
\]