Math 407A: Linear Optimization

Lecture 9
The Fundamental Theorem of Linear Programming
The Strong Duality Theorem
Complementary Slackness

Math Dept, University of Washington

1. The Two Phase Simples Algorithm

2. The Fundamental Theorem of linear Programming

3. Duality Theory Revisited

4. Complementary Slackness

The Two Phase Simplex Algorithm

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(i) The LP is determined to be unbounded.

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Duality Theory

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & c^T x \\ & \text{subject to} & Ax \leq b, \ 0 \leq x \end{array} \qquad \begin{array}{lll} \mathcal{D} & \text{minimize} & b^T y \\ & \text{subject to} & A^T y \geq c, \ 0 \leq y \end{array}$$

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What is the dual to the dual?

The Dual of the Dual

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& 0 \leq y
\end{array}
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The Dual of the Dual

$$\begin{array}{lll} \text{minimize} & b^T y & \text{Standard} \\ \text{subject to} & A^T y \geq c, & \Longrightarrow \\ & 0 \leq y & \text{form} \end{array}$$

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The dual of the dual is the primal.

The Weak Duality Theorem

Theorem:
If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \leq y^T Ax \leq b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

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We combine the Weak Duality Theorem with the Fundamental Theorem of Linear Programming to obtain the *Strong Duality Theorem*.

The Strong Duality Theorem

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If either $\mathcal{P}$ or $\mathcal{D}$ has a finite optimal value, then so does the other, the optimal values coincide, and optimal solutions to both $\mathcal{P}$ and $\mathcal{D}$ exist.

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$$\min f(x) = e^x$$

The optimal value is zero, but no solution exists.

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The optimal tableau is

$$\left[\begin{array}{cc|c} RA & R & Rb \\ \hline c^T - y^T A & -y^T & -y^T b \end{array}\right],$$

where we have already seen that y solves $\mathcal{D}$, and the optimal values coincide.

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This concludes the proof.

Complementary Slackness

Theorem: [WDT]
If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then

$$c^T x \le y^T A x \le b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is necessarily infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is necessarily infeasible. Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

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The SDT implies that x solves $\mathcal{P}$ and y solves $\mathcal{D}$ if and only if (x, y) is a $\mathcal{P}$-$\mathcal{D}$ feasible pair and

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We now examine the consequence of this equivalence.

Complementary Slackness

The equation $c^T x = y^T Ax$ implies that

$$0 = x^T(A^T y - c) = \sum_{j=1}^{n} x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j). \qquad (\clubsuit)$$

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$\mathcal{P}$-$\mathcal{D}$ feasibility gives

$$0 \le x_j \qquad \text{and} \qquad 0 \le \sum_{i=1}^{m} a_{ij}y_i - c_j \qquad \text{for } j = 1, \ldots, n.$$

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Hence, ($\clubsuit$) can only hold if

$$x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j) = 0 \quad \text{for} \quad j = 1, \ldots, n, \quad \text{or equivalently,}$$

$$x_j = 0 \qquad \text{or} \qquad \sum_{i=1}^{m} a_{ij}y_i = c_j \qquad \text{or both for } j = 1, \ldots, n.$$

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Similarly, the equation $y^T Ax = b^T y$ implies that

$$0 = y^T(b - Ax) = \sum_{i=1}^{m} y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j).$$

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Therefore, $\quad y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j) = 0 \quad i = 1, 2, \dots, m.$

Hence,

$$y_i = 0 \qquad \text{or} \qquad \sum_{j=1}^{n} a_{ij}x_j = b_i \qquad \text{or both for } i = 1, \dots, m.$$

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- $y_i = 0 \quad \text{or} \quad \sum_{j=1}^n a_{ij} x_j = b_i \quad$ or both for $i = 1, \ldots, m$.

Complementary Slackness Theorem

Theorem:
The vector $x \in \mathbb{R}^n$ solves $\mathcal{P}$ and the vector $y \in \mathbb{R}^m$ solves $\mathcal{D}$ if and only if x is feasible for $\mathcal{P}$ and y is feasible for $\mathcal{D}$ and

(i) *either* $0 = x_j$ *or* $\sum_{i=1}^{m} a_{ij} y_i = c_j$ *or both for* $j = 1, \ldots, n$, *and*

(ii) *either* $0 = y_i$ *or* $\sum_{j=1}^{n} a_{ij} x_j = b_i$ *or both for* $i = 1, \ldots, m$.

Corollary to the Complementary Slackness Theorem

Corollary:
The vector $x \in \mathbb{R}^n$ solves $\mathcal{P}$ if and only if x is feasible for $\mathcal{P}$ and there exists a vector $y \in \mathbb{R}^m$ feasible for $\mathcal{D}$ and such that

(i) *if* $\sum_{j=1}^{n} a_{ij} x_j < b$, *then* $y_i = 0$, *for* $i = 1, \ldots, m$ *and*

(ii) *if* $0 < x_j$, *then* $\sum_{i=1}^{m} a_{ij} y_i = c_j$, *for* $j = 1, \ldots, n$.

Testing Optimality via Complementary Slackness

Does

$$x = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$$

solve the LP

$$\begin{array}{lrcrcrcrcrcl}
\text{maximize} & 7x_1 & + & 6x_2 & + & 5x_3 & - & 2x_4 & + & 3x_5 & & \\
\text{subject to} & x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \leq & 4 \\
 & 4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \leq & 3 \\
 & 2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \leq & 5 \\
 & 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \leq & 1 \\
\end{array}$$
$$0 \leq x_1, x_2, x_3, x_4, x_5.$$

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& 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \leq & 1 & : & \color{red}{y_4}
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$$0 \leq x_1, x_2, x_3, x_4, x_5.$$

Testing Optimality via Complementary Slackness

The point

$$x = (x_1, x_2, x_3, x_4, x_5) = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$$

must be feasible for the LP.

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must be feasible for the LP.
Plugging into the constraints we get

$$\begin{array}{rcrcrcrcrcl}
(0) & + & 3\left(\frac{4}{3}\right) & + & 5\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 2(0) & = & 4 \\
4(0) & + & 2\left(\frac{4}{3}\right) & - & 2\left(\frac{2}{3}\right) & + & \left(\frac{5}{3}\right) & + & (0) & = & 3 \\
2(0) & + & 4\left(\frac{4}{3}\right) & + & 4\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 5(0) & < & 5 \\
3(0) & + & \left(\frac{4}{3}\right) & + & 2\left(\frac{2}{3}\right) & - & \left(\frac{5}{3}\right) & - & 2(0) & = & 1.
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\end{array}$$

Can we use this information to construct a solution to the dual problem, (y_1, y_2, y_3, y_4)?

Testing Optimality via Complementary Slackness

Recall that
if $\sum_{j=1}^{n} a_{ij}x_j < b$, *then* $y_i = 0$, *for* $i = 1, \ldots, m$.

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Testing Optimality via Complementary Slackness

Recall that
if $\sum_{j=1}^{n} a_{ij}x_j < b$, *then* $y_i = 0$, *for* $i = 1, \ldots, m$.
We have just showed that

$$
\begin{array}{rcrcrcrcrcrcl}
(0) & + & 3\left(\frac{4}{3}\right) & + & 5\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 2(0) & = & 4 & : & \color{red}{y_1} \\
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2(0) & + & 4\left(\frac{4}{3}\right) & + & 4\left(\frac{2}{3}\right) & - & 2\left(\frac{5}{3}\right) & + & 5(0) & < & 5 & : & y_3 = 0 \\
3(0) & + & \left(\frac{4}{3}\right) & + & 2\left(\frac{2}{3}\right) & - & \left(\frac{5}{3}\right) & - & 2(0) & = & 1 & : & y_4
\end{array}$$

Testing Optimality via Complementary Slackness

Also recall that
if $0 < x_j$, *then* $\sum_{i=1}^{m} a_{ij} y_i = c_j$, *for* $j = 1, \ldots, n$.

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Hence,

$$3y_1 + 2y_2 + 4y_3 + y_4 = 6 \qquad (x_2 = \tfrac{4}{3} > 0)$$

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if $0 < x_j$, *then* $\sum_{i=1}^{m} a_{ij} y_i = c_j$, *for* $j = 1, \ldots, n$.

Hence,

$$\begin{array}{rcrcrcrcrl} 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & = & 6 & \qquad (x_2 = \frac{4}{3} > 0) \\ 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & = & 5 & \qquad (x_3 = \frac{2}{3} > 0) \end{array}$$

Testing Optimality via Complementary Slackness

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Hence,

$$\begin{array}{rcrcrcrcrl}
3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & = & 6 & \quad (x_2 = \frac{4}{3} > 0) \\
5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & = & 5 & \quad (x_3 = \frac{2}{3} > 0) \\
-2y_1 & + & y_2 & - & 2y_3 & - & y_4 & = & -2 & \quad (x_4 = \frac{5}{3} > 0)
\end{array}$$

Testing Optimality via Complementary Slackness

Combining these observations gives the system

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 5 & -2 & 4 & 2 \\ -2 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -2 \\ 0 \end{pmatrix},$$

which any dual solution must satisfy.

Testing Optimality via Complementary Slackness

Combining these observations gives the system

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 5 & -2 & 4 & 2 \\ -2 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -2 \\ 0 \end{pmatrix},$$

which any dual solution must satisfy.
This is a square system that we can try to solve for y.

Testing Optimality via Complementary Slackness

$$
\begin{array}{rrrr|rl}
\hline
3 & 2 & 4 & 1 & 6 & \\
5 & -2 & 4 & 2 & 5 & \\
-2 & 1 & -2 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
3 & 2 & 0 & 1 & 6 & \quad r_1 - 4r_4 \\
5 & -2 & 0 & 2 & 5 & \quad r_2 - 4r_4 \\
-2 & 1 & 0 & -1 & -2 & \quad r_3 + 2r_4 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 3 & 0 & 0 & 4 & \quad r_1 + r_3 \\
1 & 0 & 0 & 0 & 1 & \quad r_2 + 2r_3 \\
-2 & 1 & 0 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
\end{array}
$$

Testing Optimality via Complementary Slackness

$$
\begin{array}{rrrr|r l}
\hline
3 & 2 & 4 & 1 & 6 & \\
5 & -2 & 4 & 2 & 5 & \\
-2 & 1 & -2 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
3 & 2 & 0 & 1 & 6 & r_1 - 4r_4 \\
5 & -2 & 0 & 2 & 5 & r_2 - 4r_4 \\
-2 & 1 & 0 & -1 & -2 & r_3 + 2r_4 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 3 & 0 & 0 & 4 & r_1 + r_3 \\
1 & 0 & 0 & 0 & 1 & r_2 + 2r_3 \\
-2 & 1 & 0 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
\end{array}
$$

$$
\begin{array}{rrrr|r l}
\hline
1 & 3 & 0 & 0 & 4 & r_1 + r_3 \\
1 & 0 & 0 & 0 & 1 & r_2 + 2r_3 \\
-2 & 1 & 0 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
0 & 3 & 0 & 0 & 3 & r_1 - r_2 \\
1 & 0 & 0 & 0 & 1 & \\
0 & 1 & 0 & -1 & 0 & r_3 + 2r_2 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & 1 & r_2 \\
0 & 1 & 0 & 0 & 1 & \frac{1}{3}r_1 \\
0 & 0 & 1 & 0 & 0 & r_4 \\
0 & 0 & 0 & 1 & 1 & -r_3 + \frac{1}{3}r_1 \\
\hline
\end{array}
$$

Testing Optimality via Complementary Slackness

$$
\begin{array}{rrrr|rl}
\hline
3 & 2 & 4 & 1 & 6 & \\
5 & -2 & 4 & 2 & 5 & \\
-2 & 1 & -2 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
3 & 2 & 0 & 1 & 6 & r_1 - 4r_4 \\
5 & -2 & 0 & 2 & 5 & r_2 - 4r_4 \\
-2 & 1 & 0 & -1 & -2 & r_3 + 2r_4 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 3 & 0 & 0 & 4 & r_1 + r_3 \\
1 & 0 & 0 & 0 & 1 & r_2 + 2r_3 \\
-2 & 1 & 0 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
\end{array}
$$

$$
\begin{array}{rrrr|rl}
\hline
1 & 3 & 0 & 0 & 4 & r_1 + r_3 \\
1 & 0 & 0 & 0 & 1 & r_2 + 2r_3 \\
-2 & 1 & 0 & -1 & -2 & \\
0 & 0 & 1 & 0 & 0 & \\
\hline
0 & 3 & 0 & 0 & 3 & r_1 - r_2 \\
1 & 0 & 0 & 0 & 1 & \\
0 & 1 & 0 & -1 & 0 & r_3 + 2r_2 \\
0 & 0 & 1 & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & 1 & r_2 \\
0 & 1 & 0 & 0 & 1 & \frac{1}{3}r_1 \\
0 & 0 & 1 & 0 & 0 & r_4 \\
0 & 0 & 0 & 1 & 1 & -r_3 + \frac{1}{3}r_1 \\
\hline
\end{array}
$$

This gives the solution $(y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$.

Testing Optimality via Complementary Slackness

$$
\begin{array}{rrrr|r}
\hline
3 & 2 & 4 & 1 & 6 \\
5 & -2 & 4 & 2 & 5 \\
-2 & 1 & -2 & -1 & -2 \\
0 & 0 & 1 & 0 & 0 \\
\hline
3 & 2 & 0 & 1 & 6 \\
5 & -2 & 0 & 2 & 5 \\
-2 & 1 & 0 & -1 & -2 \\
0 & 0 & 1 & 0 & 0 \\
\hline
1 & 3 & 0 & 0 & 4 \\
1 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -1 & -2 \\
0 & 0 & 1 & 0 & 0 \\
\hline
\end{array}
\quad
\begin{array}{l}
\\ \\ \\ \\
r_1 - 4r_4 \\
r_2 - 4r_4 \\
r_3 + 2r_4 \\
\\
r_1 + r_3 \\
r_2 + 2r_3 \\
\\
\\
\end{array}
\qquad
\begin{array}{rrrr|r}
\hline
1 & 3 & 0 & 0 & 4 \\
1 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -1 & -2 \\
0 & 0 & 1 & 0 & 0 \\
\hline
0 & 3 & 0 & 0 & 3 \\
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
\quad
\begin{array}{l}
r_1 + r_3 \\
r_2 + 2r_3 \\
\\
\\
r_1 - r_2 \\
\\
r_3 + 2r_2 \\
\\
r_2 \\
\frac{1}{3}r_1 \\
r_4 \\
-r_3 + \frac{1}{3}r_1
\end{array}
$$

This gives the solution $(y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$.

Is this dual feasible?

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
& 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
& 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
& -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
& 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
& & & \multicolumn{5}{c}{0 \leq y_1, y_2, y_3, y_4.} & &
\end{array}
$$

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & \multicolumn{5}{c}{0 \leq y_1, y_2, y_3, y_4.} & &
\end{array}$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
& 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
& 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
& -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
& 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
& & & & & \multicolumn{5}{l}{0 \leq y_1, y_2, y_3, y_4.}
\end{array}$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

Testing Optimality via Complementary Slackness

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$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & & 0 \leq y_1, y_2, y_3, y_4. & & & & &
\end{array}
$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

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First: $1 + 4 + 0 + 3 = 8 > 7$

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & \multicolumn{5}{c}{0 \leq y_1, y_2, y_3, y_4.} & &
\end{array}$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

First: $1 + 4 + 0 + 3 = 8 > 7$

Fifth: $2 + 1 + 0 - 2 = 1 \not\geq 3$, the fifth dual inequality is violated.

Testing Optimality via Complementary Slackness

$y = (y_1, y_2, y_3, y_4) = (1,\ 1,\ 0,\ 1)$

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{subject to} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & & & 0 \leq y_1, y_2, y_3, y_4. & & & &
\end{array}
$$

Clearly, $0 \leq y$ and by construction the 2nd, 3rd, and 4th of the linear inequality constraints are satisfied with equality.

We need to check the first and inequalities.

First: $1 + 4 + 0 + 3 = 8 > 7$

Fifth: $2 + 1 + 0 - 2 = 1 \not\geq 3$, the fifth dual inequality is violated.

Hence, $x = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$ cannot be optimal!

Example: Testing Optimality via Complementary Slackness

Does the point $x = (1, 1, 1, 0)$ solve the following LP?

$$
\begin{array}{lrrrrcl}
\text{maximize} & 4x_1 & +2x_2 & +2x_3 & +4x_4 & & \\
\text{subject to} & x_1 & +3x_2 & +2x_3 & + \; x_4 & \leq & 7 \\
 & x_1 & + \; x_2 & + \; x_3 & +2x_4 & \leq & 3 \\
 & 2x_1 & & + \; x_3 & + \; x_4 & \leq & 3 \\
 & x_1 & + \; x_2 & & +2x_4 & \leq & 2 \\
 & 0 \leq & x_1, x_2, & x_3, x_4 & & &
\end{array}
$$