

Math 407A: Linear Optimization

Lecture 8: Initialization and the Two Phase Simplex Algorithm

Math Dept, University of Washington

1 Initialization

2 The Auxilliary Problem

3 The Two Phase Simplex Algorithm

Initialization

We have shown that if we are given a feasible dictionary (tableau) for an LP, then the simplex algorithm will terminate finitely if it is employed with a anti-cycling rule.

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We now address the question of how to determine an initial feasible dictionary (tableau).

The Auxiliary Problem

$$\begin{array}{ll} \mathcal{P} & \text{maximize} \quad c^T x \\ & \text{subject to} \quad Ax \leq b, \quad 0 \leq x. \end{array}$$

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Consider an auxiliary LP of the form

$$\begin{array}{ll} \mathcal{Q} & \text{minimize} \quad x_0 \\ & \text{subject to} \quad Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x. \end{array}$$

where $\mathbf{1} \in \mathbb{R}^m$ is the vector of all ones.

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where $\mathbf{1} \in \mathbb{R}^m$ is the vector of all ones.

The i^{th} row of the system of inequalities $Ax - x_0 \mathbf{1} \leq b$ is

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i + x_0.$$

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$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i + x_0.$$

In block matrix form we write

$$\begin{bmatrix} -\mathbf{1} & A \end{bmatrix} \begin{pmatrix} x_0 \\ x \end{pmatrix} \leq b.$$

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Plugging into $Ax - x_0 \mathbf{1} \leq b$, we get $A\tilde{x} \leq b$, i.e. \tilde{x} is feasible for \mathcal{P} .

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Plugging into $Ax - x_0 \mathbf{1} \leq b$, we get $A\tilde{x} \leq b$, i.e. \tilde{x} is feasible for \mathcal{P} .

On the other hand, if \hat{x} is feasible for \mathcal{P} , then (\hat{x}_0, \hat{x}) with $\hat{x}_0 = 0$ is feasible for \mathcal{Q} , so (\hat{x}_0, \hat{x}) is optimal for \mathcal{Q} .

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- \mathcal{P} is feasible \Leftrightarrow the optimal value in \mathcal{Q} is zero.

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- \mathcal{P} is feasible \Leftrightarrow the optimal value in \mathcal{Q} is zero.
- \mathcal{P} is infeasible \Leftrightarrow the optimal value in \mathcal{Q} is positive.

Two Phase Simplex Algorithm

The auxiliary problem Q is also called the *Phase I* problem since solving it is the first phase of a two phase process of solving general LPs.

Two Phase Simplex Algorithm

The auxiliary problem \mathcal{Q} is also called the *Phase I* problem since solving it is the first phase of a two phase process of solving general LPs.

In Phase I we solve the auxiliary problem to obtain an initial feasible tableau for \mathcal{P} , and in Phase II we solve the original LP starting with the feasible tableau provided in Phase I.

Initializing the Auxiliary Problem

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Problem: The initial dictionary for \mathcal{Q} is infeasible!

Solution: Set $x_0 = -\min\{b_i : i = 0, \dots, n\}$ with $b_0 = 0$,
then $b + x_0 \mathbf{1} \geq 0$ since

$$\begin{aligned} \min\{b_i + x_0 : i = 1, \dots, m\} &= \min\{b_i : i = 1, \dots, m\} + x_0 \\ &= \min\{b_i : i = 1, \dots, m\} - \min\{b_i : i = 0, \dots, m\} \geq 0. \end{aligned}$$

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Hence, $x_0 = -\min\{b_i : i = 0, \dots, m\}$ and $x = 0$ is feasible for \mathcal{Q} .

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Hence, $x_0 = -\min\{b_i : i = 0, \dots, m\}$ and $x = 0$ is feasible for \mathcal{Q} .
It is also a BFS for \mathcal{Q} .

Initializing the Auxiliary Problem

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The initial dictionary for \mathcal{Q} is

$$\begin{array}{rcl} x_{n+i} & = & b_i + x_0 - \sum_{j=1}^m a_{ij} x_j \\ z & = & -x_0. \end{array}$$

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Let i_0 be such that

$$b_{i_0} = \min\{b_i : i = 0, 1, \dots, m\}.$$

If $i_0 = 0$, the LP has feasible origin and so the initial dictionary is optimal.

Initializing the Auxiliary Problem

If $i_0 > 0$, then pivot on this row bringing x_0 into the basis yielding

$$x_0 = -b_{i_0} + x_{n+i_0} + \sum_{j=1}^m a_{i_0j} x_j$$

$$x_{n+i} = b_i - b_{i_0} + x_{n+i_0} - \sum_{j=1}^m (a_{ij} - a_{i_0j}) x_j, \quad i \neq i_0$$

$$z = b_{i_0} - x_{n+i_0} - \sum_{j=1}^m a_{i_0j} x_j.$$

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$$z = b_{i_0} - x_{n+i_0} - \sum_{j=1}^m a_{i_0j} x_j.$$

This dictionary is feasible for \mathcal{Q} .

Initializing the Auxiliary Problem: Example

$$\begin{array}{ll} \max & x_1 - x_2 + x_3 \\ \text{s.t.} & 2x_1 - x_2 + 2x_3 \leq 4 \\ & 2x_1 - 3x_2 + x_3 \leq -5 \\ & -x_1 + x_2 - 2x_3 \leq -1 \\ & 0 \leq x_1, x_2, x_3 . \end{array}$$

Initializing the Auxiliary Problem: Example

$$\begin{array}{llllll} \max & x_1 & - & x_2 & + & x_3 \\ \text{s.t.} & 2x_1 & - & x_2 & + & 2x_3 \leq 4 \\ & 2x_1 & - & 3x_2 & + & x_3 \leq -5 \\ & -x_1 & + & x_2 & - & 2x_3 \leq -1 \\ & & & & & 0 \leq x_1, x_2, x_3 . \end{array}$$

$$\begin{array}{llllll} \max & -x_0 \\ \text{s.t.} & -x_0 & + & 2x_1 & - & x_2 & + & 2x_3 \leq 4 \\ & -x_0 & + & 2x_1 & - & 3x_2 & + & x_3 \leq -5 \\ & -x_0 & - & x_1 & + & x_2 & - & 2x_3 \leq -1 \\ & & & & & & & 0 \leq x_0, x_1, x_2, x_3 . \end{array}$$

Example

$$\begin{aligned} \max \quad & -x_0 \\ \text{s.t.} \quad & -x_0 + 2x_1 - x_2 + 2x_3 \leq 4 \\ & -x_0 + 2x_1 - 3x_2 + x_3 \leq -5 \\ & -x_0 - x_1 + x_2 - 2x_3 \leq -1 \\ & 0 \leq x_0, x_1, x_2, x_3 . \end{aligned}$$

Example

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$$\begin{array}{ccccccc|c} -1 & 2 & -1 & 2 & 1 & 0 & 0 & 4 \\ -1 & 2 & -3 & 1 & 0 & 1 & 0 & -5 \\ -1 & -1 & 1 & -2 & 0 & 0 & 1 & -1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

First Pivot

	x_0							
	-1	2	-1	2	1	0	0	4
	-1	2	-3	1	0	1	0	-5
	-1	-1	1	-2	0	0	1	-1
z	0	1	-1	1	0	0	0	0
w	-1	0	0	0	0	0	0	0

First Pivot

	x_0								
	-1	2	-1	2	1	0	0	4	
	$\ominus 1$	2	-3	1	0	1	0	-5	most negative
	-1	-1	1	-2	0	0	1	-1	
z	0	1	-1	1	0	0	0	0	
w	-1	0	0	0	0	0	0	0	

First Pivot

	x_0							
	-1	2	-1	2	1	0	0	4
	\ominus 1	2	-3	1	0	1	0	-5
	-1	-1	1	-2	0	0	1	-1
z	0	1	-1	1	0	0	0	0
w	-1	0	0	0	0	0	0	0
	0	0	2	1	1	-1	0	9
	1	-2	3	-1	0	-1	0	5
	0	-3	4	-3	0	-1	1	4
z	0	1	-1	1	0	0	0	0
w	0	-2	3	-1	0	-1	0	5

First Pivot

	x_0							
	-1	2	-1	2	1	0	0	4
	-1	2	-3	1	0	1	0	-5
	-1	-1	1	-2	0	0	1	-1
z	0	1	-1	1	0	0	0	0
w	-1	0	0	0	0	0	0	0
	0	0	2	1	1	-1	0	9
	1	-2	3	-1	0	-1	0	5
	0	-3	④	-3	0	-1	1	4
z	0	1	-1	1	0	0	0	0
w	0	-2	3	-1	0	-1	0	5

Second Pivot

	0	0	2	1	1	-1	0	9
	1	-2	3	-1	0	-1	0	5
	0	-3	4	-3	0	-1	1	4
z	0	1	-1	1	0	0	0	0
w	0	-2	3	-1	0	-1	0	5

Second Pivot

	0	0	2	1	1	-1	0	9
	1	-2	3	-1	0	-1	0	5
	0	-3	4	-3	0	-1	1	4
z	0	1	-1	1	0	0	0	0
w	0	-2	3	-1	0	-1	0	5
	0	$\frac{3}{2}$	0	$\frac{5}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	7
	1	$\frac{1}{4}$	0	$\frac{5}{4}$	0	$-\frac{1}{4}$	$-\frac{3}{4}$	2
	0	$-\frac{3}{4}$	1	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	1
z	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	1
w	0	$\frac{1}{4}$	0	$\frac{5}{4}$	0	$-\frac{1}{4}$	$-\frac{3}{4}$	2

Second Pivot

	0	0	2	1	1	-1	0	9
	1	-2	3	-1	0	-1	0	5
	0	-3	4	-3	0	-1	1	4
z	0	1	-1	1	0	0	0	0
w	0	-2	3	-1	0	-1	0	5
	0	$\frac{3}{2}$	0	$\frac{5}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	7
	1	$\frac{1}{4}$	0	$\frac{5}{4}$	0	$-\frac{1}{4}$	$-\frac{3}{4}$	2
	0	$-\frac{3}{4}$	1	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	1
z	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	1
w	0	$\frac{1}{4}$	0	$\frac{5}{4}$	0	$-\frac{1}{4}$	$-\frac{3}{4}$	2

Third Pivot

	0	$\frac{3}{2}$	0	$\frac{5}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	7
	1	$\frac{1}{4}$	0	$\frac{5}{4}$	0	$-\frac{1}{4}$	$-\frac{3}{4}$	2
	0	$-\frac{3}{4}$	1	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	1
z	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	1
w	0	$\frac{1}{4}$	0	$\frac{5}{4}$	0	$-\frac{1}{4}$	$-\frac{3}{4}$	2
	-2	1	0	0	1	0	1	3
	$\frac{4}{5}$	$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$
	$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$
z	$-\frac{1}{5}$	$\frac{4}{20}$	0	0	0	$-\frac{4}{20}$	$\frac{8}{20}$	$\frac{3}{5}$
w	-1	0	0	0	0	0	0	0

Third Pivot

	0	$\frac{3}{2}$	0	$\frac{5}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	7	
	1	$\frac{1}{4}$	0	$\frac{5}{4}$	0	$-\frac{1}{4}$	$-\frac{3}{4}$	2	
	0	$-\frac{3}{4}$	1	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	1	
z	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	1	
w	0	$\frac{1}{4}$	0	$\frac{5}{4}$	0	$-\frac{1}{4}$	$-\frac{3}{4}$	2	
	-2	1	0	0	1	0	1	3	Auxiliary
	$\frac{4}{5}$	$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$	problem
	$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$	solved.
z	$-\frac{1}{5}$	$\frac{4}{20}$	0	0	0	$-\frac{4}{20}$	$\frac{8}{20}$	$\frac{3}{5}$	
w	-1	0	0	0	0	0	0	0	

Auxiliary Problem Solution

	-2	1	0	0	1	0	1	3	
	$\frac{4}{5}$	$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$	
	$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$	
z	$-\frac{1}{5}$	$\frac{4}{20}$	0	0	0	$-\frac{4}{20}$	$\frac{8}{20}$	$\frac{3}{5}$	Original LP
w	-1	0	0	0	0	0	0	0	is feasible.

Extract Initial Feasible Tableau

	-2	1	0	0	1	0	1	3	Extract initial. feasible tableau.
	$\frac{4}{5}$	$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$	
	$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$	
z	$-\frac{1}{5}$	$\frac{4}{20}$	0	0	0	$-\frac{4}{20}$	$\frac{8}{20}$	$\frac{3}{5}$	
w	-1	0	0	0	0	0	0	0	

Third Pivot

	-2	1	0	0	1	0	1	3	Extract initial. feasible tableau.
	$\frac{4}{5}$	$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$	
	$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$	
z	$-\frac{1}{5}$	$\frac{4}{20}$	0	0	0	$-\frac{4}{20}$	$\frac{8}{20}$	$\frac{3}{5}$	
w	-1	0	0	0	0	0	0	0	
	-2	1	0	0	1	0	1	3	
	$\frac{4}{5}$	$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$	
	$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$	
z	$-\frac{1}{5}$	$\frac{4}{20}$	0	0	0	$-\frac{4}{20}$	$\frac{8}{20}$	$\frac{3}{5}$	
w	-1	0	0	0	0	0	0	0	

Phase II

1	0	0	1	0	①	3
$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$
$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$
$\frac{1}{5}$	0	0	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$
1	0	0	1	0	1	3
$\frac{4}{5}$	0	1	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{17}{5}$
$-\frac{2}{5}$	1	0	$\frac{1}{5}$	0	0	$\frac{14}{5}$
$-\frac{1}{5}$	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	$-\frac{3}{5}$

Phase II: Solution

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ \frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\ -\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\ \hline -\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5} \end{array}$$

Phase II: Solution

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ \frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\ -\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\ \hline -\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5} \end{array}$$

Optimal primal and dual solutions are

Phase II: Solution

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ \frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\ -\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\ \hline -\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5} \end{array}$$

Optimal primal and dual solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.8 \\ 3.4 \end{pmatrix}$$

Phase II: Solution

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ \frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\ -\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\ \hline -\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5} \end{array}$$

Optimal primal and dual solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.8 \\ 3.4 \end{pmatrix} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0 \end{pmatrix}$$

Phase II: Solution

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ \frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\ -\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\ \hline -\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5} \end{array}$$

Optimal primal and dual solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.8 \\ 3.4 \end{pmatrix} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0 \end{pmatrix}$$

with optimal value $z = .6$.

Example: Two Phase Simplex Algorithm

Use the two phase simplex method to solve the following LP:

$$\begin{array}{llllll} \text{maximize} & 3x_1 & + & x_2 & & \\ \text{subject to} & x_1 & - & x_2 & \leq & -1 \\ & -x_1 & - & x_2 & \leq & -3 \\ & 2x_1 & + & x_2 & \leq & 4 \\ & & & & 0 & \leq & x_1, & x_2 \end{array}$$

Hint: A complete solution is possible in 3 pivots.