Math 407A: Linear Optimization

Lecture 8: Initialization and the Two Phase Simplex Algorithm

Math Dept, University of Washington

Initialization

We have shown that if we are given a feasible dictionary (tableau) for an LP, then the simplex algorithm will terminate finitely if it is employed with a anti-cycling rule.

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The simplex algorithm will terminate in one of two ways:

- The LP is determined to be unbounded.
- An optimal BFS is found.

We now address the question of how to determine an initial feasible dictionary (tableau).

The Auxiliary Problem

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & c^T x \\ & \text{subject tp} & Ax \leq b, \quad 0 \leq x. \end{array}$$

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Consider an auxiliary LP of the form

$$\begin{array}{lll} \mathcal{Q} & \text{minimize} & x_0 \\ & \text{subject to} & Ax - x_0\mathbf{1} \leq b, \quad 0 \leq x_0, x \ . \end{array}$$

where $\mathbf{1} \in \mathbb{R}^m$ is the vector of all ones.

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where $\mathbf{1} \in \mathbb{R}^m$ is the vector of all ones.
The ith row of the system of inequalities $Ax - x_0\mathbf{1} \leq b$ is

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i + x_0 \ .$$

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$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i + x_0 \ .$$

In block matrix form we write

$$\begin{bmatrix} -\mathbf{1} & A \end{bmatrix} \begin{pmatrix} x_0 \\ x \end{pmatrix} \le b \ .$$

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If the optimal value in the auxiliary problem is zero, then at the optimal solution $(\tilde{x}_0, \tilde{x})$ we have $\tilde{x}_0 = 0$.

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Plugging into $Ax - x_0\mathbf{1} \le b$, we get $A\tilde{x} \le b$, i.e. $\tilde{x}$ is feasible for $\mathcal{P}$.

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Plugging into $Ax - x_0\mathbf{1} \le b$, we get $A\tilde{x} \le b$, i.e. $\tilde{x}$ is feasible for $\mathcal{P}$.

On the other hand, if $\hat{x}$ is feasible for $\mathcal{P}$, then $(\hat{x}_0, \hat{x})$ with $\hat{x}_0 = 0$ is feasible for $\mathcal{Q}$, so $(\hat{x}_0, \hat{x})$ is optimal for $\mathcal{Q}$.

The Auxiliary Problem

$$\begin{array}{llll} \mathcal{P} & \text{maximize} & c^T x \\ & \text{subject tp} & Ax \le b, \\ & & 0 \le x \end{array} \qquad \begin{array}{llll} \mathcal{Q} & \text{minimize} & x_0 \\ & \text{subject to} & Ax - x_0\mathbf{1} \le b, \\ & & 0 \le x_0, x \end{array}$$

The Auxiliary Problem

$$
\begin{array}{llll}
\mathcal{P} & \text{maximize} & c^T x \\
& \text{subject tp} & Ax \leq b, \\
& & 0 \leq x
\end{array}
\qquad
\begin{array}{llll}
\mathcal{Q} & \text{minimize} & x_0 \\
& \text{subject to} & Ax - x_0\mathbf{1} \leq b, \\
& & 0 \leq x_0, x
\end{array}
$$

- $\mathcal{P}$ is feasible $\Leftrightarrow$ the optimal value in $\mathcal{Q}$ is zero.

The Auxiliary Problem

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & c^T x \\ & \text{subject tp} & Ax \le b, \\ & & 0 \le x \end{array} \qquad \begin{array}{lll} \mathcal{Q} & \text{minimize} & x_0 \\ & \text{subject to} & Ax - x_0\mathbf{1} \le b, \\ & & 0 \le x_0, x \end{array}$$

- $\mathcal{P}$ is feasible $\Leftrightarrow$ the optimal value in $\mathcal{Q}$ is zero.
- $\mathcal{P}$ is infeasible $\Leftrightarrow$ the optimal value in $\mathcal{Q}$ is positive.

Two Phase Simplex Algorithm

The auxiliary problem $\mathcal{Q}$ is also called the *Phase I* problem since solving it is the first phase of a two phase process of solving general LPs.

Two Phase Simplex Algorithm

The auxiliary problem $\mathcal{Q}$ is also called the *Phase I* problem since solving it is the first phase of a two phase process of solving general LPs.

In Phase I we solve the auxiliary problem to obtain an initial feasible tableau for $\mathcal{P}$, and in Phase II we solve the original LP starting with the feasible tableau provided in Phase I.

Initializing the Auxiliary Problem

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Initializing the Auxiliary Problem

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Problem: The initial dictionary for $\mathcal{Q}$ is infeasible!

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Problem: The initial dictionary for $\mathcal{Q}$ is infeasible!

Solution: Set $x_0 = -\min\{b_i : i = 0, \ldots, n\}$ with $b_0 = 0$,
then $b + x_0\mathbf{1} \ge 0$ since

$$\begin{aligned} \min\{b_i + x_0 : i = 1, \ldots, m\} &= \min\{b_i : i = 1, \ldots, m\} + x_0 \\ = \min\{b_i : i = 1, \ldots, m\} &- \min\{b_i : i = 0, \ldots, m\} \ge 0. \end{aligned}$$

Initializing the Auxiliary Problem

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Hence, $x_0 = -\min\{b_i : i = 0, \ldots, m\}$ and $x = 0$ is feasible for $\mathcal{Q}$.

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Hence, $x_0 = -\min\{b_i : i = 0, \dots, m\}$ and $x = 0$ is feasible for $\mathcal{Q}$.
It is also a BFS for $\mathcal{Q}$.

Initializing the Auxiliary Problem

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The initial dictionary for $\mathcal{Q}$ is

$$\begin{aligned} x_{n+i} &= b_i + x_0 - \sum_{j=1}^{m} a_{ij}x_j \\ z &= -x_0. \end{aligned}$$

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$$b_{i_0} = \min\{b_i : i = 0, 1, \ldots, m).$$

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If $i_0 = 0$,

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Let i_0 be such that

$$b_{i_0} = \min\{b_i : i = 0, 1, \ldots, m).$$

If $i_0 = 0$, the LP has feasible origin and so the initial dictionary is optimal.

Initializing the Auxiliary Problem

If $i_0 > 0$, then pivot on this row bringing x_0 into the basis yielding

$$\begin{aligned}
x_0 &= -b_{i_0} + x_{n+i_0} + \sum_{j=1}^{m} a_{i_0 j} x_j \\
x_{n+i} &= b_i - b_{i_0} + x_{n+i_0} - \sum_{j=1}^{m} (a_{ij} - a_{i_0 j}) x_j, \quad i \neq i_0 \\
z &= b_{i_0} - x_{n+i_0} - \sum_{j=1}^{m} a_{i_0 j} x_j.
\end{aligned}$$

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z &= b_{i_0} - x_{n+i_0} - \sum_{j=1}^{m} a_{i_0 j} x_j.
\end{aligned}$$

This dictionary is feasible for $\mathcal{Q}$.

Initializing the Auxiliary Problem: Example

$$
\begin{array}{lrcrcrcr}
\max & x_1 & - & x_2 & + & x_3 & & \\
\text{s.t.} & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
 & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
 & -x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
\end{array}
$$

$$0 \leq x_1, x_2, x_3 \; .$$

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\text{s.t.} & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
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 & -x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
\end{array}
$$

$$0 \leq x_1, x_2, x_3 \ .$$

$$
\begin{array}{lrcrcrcrcr}
\max & -x_0 & & & & & & & & \\
\text{s.t.} & -x_0 & + & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
 & -x_0 & + & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
 & -x_0 & - & x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
\end{array}
$$

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Example

$$
\begin{array}{llllllllllll}
\max & -x_0 \\
\text{s.t.} & -x_0 & + & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
& -x_0 & + & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
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Example

$$
\begin{array}{llcrcrcrcr}
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\text{s.t.} & -x_0 & + & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
& -x_0 & + & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
& -x_0 & - & x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
\end{array}
$$

$$0 \leq x_0, x_1, x_2, x_3 \quad .$$

$$
\begin{array}{rrrrrrr|r}
-1 & 2 & -1 & 2 & 1 & 0 & 0 & 4 \\
-1 & 2 & -3 & 1 & 0 & 1 & 0 & -5 \\
-1 & -1 & 1 & -2 & 0 & 0 & 1 & -1 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

First Pivot

	x_0							
	-1	2	-1	2	1	0	0	4
	-1	2	-3	1	0	1	0	-5
	-1	-1	1	-2	0	0	1	-1
z	0	1	-1	1	0	0	0	0
w	-1	0	0	0	0	0	0	0

First Pivot

	x_0								
	-1	2	-1	2	1	0	0	4	
	-1	2	-3	1	0	1	0	-5	most negative
	-1	-1	1	-2	0	0	1	-1	
z	0	1	-1	1	0	0	0	0	
w	-1	0	0	0	0	0	0	0	

First Pivot

	x_0							
	-1	2	-1	2	1	0	0	4
	-1	2	-3	1	0	1	0	-5
	-1	-1	1	-2	0	0	1	-1
z	0	1	-1	1	0	0	0	0
w	-1	0	0	0	0	0	0	0

	0	0	2	1	1	-1	0	9
	1	-2	3	-1	0	-1	0	5
	0	-3	4	-3	0	-1	1	4
z	0	1	-1	1	0	0	0	0
w	0	-2	3	-1	0	-1	0	5

First Pivot

	x_0							
	-1	2	-1	2	1	0	0	4
	-1	2	-3	1	0	1	0	-5
	-1	-1	1	-2	0	0	1	-1
z	0	1	-1	1	0	0	0	0
w	-1	0	0	0	0	0	0	0

	0	0	2	1	1	-1	0	9
	1	-2	3	-1	0	-1	0	5
	0	-3	4	-3	0	-1	1	4
z	0	1	-1	1	0	0	0	0
w	0	-2	3	-1	0	-1	0	5

Second Pivot

	0	0	2	1	1	-1	0	9
	1	-2	3	-1	0	-1	0	5
	0	-3	4	-3	0	-1	1	4
z	0	1	-1	1	0	0	0	0
w	0	-2	3	-1	0	-1	0	5

Second Pivot

$$
\begin{array}{c|ccccccc|c}
 & 0 & 0 & 2 & 1 & 1 & -1 & 0 & 9 \\
 & 1 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
 & 0 & -3 & \textcircled{4} & -3 & 0 & -1 & 1 & 4 \\
\hline
z & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
w & 0 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
\hline
 & 0 & \frac{3}{2} & 0 & \frac{5}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 7 \\
 & 1 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
 & 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\hline
z & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
w & 0 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
\hline
\end{array}
$$

Second Pivot

$$
\begin{array}{c|ccccccc|c}
 & 0 & 0 & 2 & 1 & 1 & -1 & 0 & 9 \\
 & 1 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
 & 0 & -3 & 4 & -3 & 0 & -1 & 1 & 4 \\
\hline
z & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
w & 0 & -2 & 3 & -1 & 0 & -1 & 0 & 5 \\
\hline
\end{array}
$$

$$
\begin{array}{c|ccccccc|c}
 & 0 & \frac{3}{2} & 0 & \frac{5}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 7 \\
 & 1 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
 & 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\hline
z & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
w & 0 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
\hline
\end{array}
$$

Third Pivot

$$
\begin{array}{c|ccccccc|c}
 & 0 & \frac{3}{2} & 0 & \frac{5}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 7 \\
 & 1 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
 & 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\hline
z & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
w & 0 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
\hline
\hline
 & -2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
 & \frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
 & \frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\
\hline
z & -\frac{1}{5} & \frac{4}{20} & 0 & 0 & 0 & -\frac{4}{20} & \frac{8}{20} & \frac{3}{5} \\
\hline
w & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Third Pivot

$$
\begin{array}{c|ccccccc|c}
 & 0 & \frac{3}{2} & 0 & \frac{5}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 7 \\
 & 1 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
 & 0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\hline
z & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
w & 0 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
\hline
\end{array}
$$

$$
\begin{array}{c|ccccccc|c}
\hline
 & -2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
 & \frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
 & \frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\
\hline
z & -\frac{1}{5} & \frac{4}{20} & 0 & 0 & 0 & -\frac{4}{20} & \frac{8}{20} & \frac{3}{5} \\
\hline
w & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Auxiliary problem solved.

Auxiliary Problem Solution

$$
\begin{array}{c|ccccccc|c}
\hline
 & -2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
 & \frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
 & \frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\
\hline
z & -\frac{1}{5} & \frac{4}{20} & 0 & 0 & 0 & -\frac{4}{20} & \frac{8}{20} & \frac{3}{5} \\
\hline
w & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Original LP is feasible.

Extract Initial Feasible Tableau

	-2	1	0	0	1	0	1	3	Extract
	$\frac{4}{5}$	$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$	initial.
	$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$	feasible
z	$-\frac{1}{5}$	$\frac{4}{20}$	0	0	0	$-\frac{4}{20}$	$\frac{8}{20}$	$\frac{3}{5}$	tableau.
w	-1	0	0	0	0	0	0	0	

Third Pivot

	-2	1	0	0	1	0	1	3
	$\frac{4}{5}$	$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$
	$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$
z	$-\frac{1}{5}$	$\frac{4}{20}$	0	0	0	$-\frac{4}{20}$	$\frac{8}{20}$	$\frac{3}{5}$
w	-1	0	0	0	0	0	0	0

Extract
initial.
feasible
tableau.

	-2	1	0	0	1	0	1	3
	$\frac{4}{5}$	$\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$
	$\frac{3}{5}$	$-\frac{3}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{11}{5}$
z	$-\frac{1}{5}$	$\frac{4}{20}$	0	0	0	$-\frac{4}{20}$	$\frac{8}{20}$	$\frac{3}{5}$
w	-1	0	0	0	0	0	0	0

Phase II

$$
\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{8}{5} \\
-\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\
\hline
\frac{1}{5} & 0 & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \frac{3}{5} \\
\hline\hline
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}
$$

Phase II: Solution

$$
\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}
$$

Phase II: Solution

$$
\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}
$$

Optimal primal and dual solutions are

Phase II: Solution

$$\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}$$

Optimal primal and dual solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.8 \\ 3.4 \end{pmatrix}$$

Phase II: Solution

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ \frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\ -\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\ \hline -\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5} \end{array}$$

Optimal primal and dual solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.8 \\ 3.4 \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0 \end{pmatrix}$$

Phase II: Solution

$$
\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
\hline
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}
$$

Optimal primal and dual solutions are

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.8 \\ 3.4 \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0 \end{pmatrix}
$$

with optimal value $z = .6$.

Example: Two Phase Simplex Algorithm

Use the two phase simplex method to solve the following LP:

$$
\begin{array}{lrcrcr}
\text{maximize} & 3x_1 & + & x_2 & & \\
\text{subject to} & x_1 & - & x_2 & \leq & -1 \\
 & -x_1 & - & x_2 & \leq & -3 \\
 & 2x_1 & + & x_2 & \leq & 4 \\
 & & 0 & \leq & x_1, & x_2
\end{array}
$$

Hint: A complete solution is possible in 3 pivots.