Lecture 8: Initialization and the Two Phase Simplex Algorithm

Math Dept, University of Washington
1 Initialization

2 The Auxiliary Problem

3 The Two Phase Simplex Algorithm
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- The LP is determined to be unbounded.
- An optimal BFS is found.
Initialization

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The simplex algorithm will terminate in one of two ways:

- The LP is determined to be unbounded.
- An optimal BFS is found.

We now address the question of how to determine an initial feasible dictionary (tableau).
The Auxiliary Problem

\[ \mathcal{P} \quad \text{maximize} \quad c^T x \]
\[ \text{subject to} \quad Ax \leq b, \quad 0 \leq x. \]
The Auxiliary Problem

\[ \mathcal{P} \quad \text{maximize} \quad c^T x \]
subject to \[ Ax \leq b, \quad 0 \leq x. \]

Consider an auxiliary LP of the form

\[ \mathcal{Q} \quad \text{minimize} \quad x_0 \]
subject to \[ Ax - x_0 1 \leq b, \quad 0 \leq x_0, x. \]

where \( 1 \in \mathbb{R}^m \) is the vector of all ones.
The Auxiliary Problem

\[ \mathcal{P} \text{ maximize } c^T x \]

subject to \( Ax \leq b, \ 0 \leq x. \)

Consider an auxiliary LP of the form

\[ \mathcal{Q} \text{ minimize } x_0 \]

subject to \( Ax - x_0 \mathbf{1} \leq b, \ 0 \leq x_0, x. \)

where \( \mathbf{1} \in \mathbb{R}^m \) is the vector of all ones.

The \( i^{th} \) row of the system of inequalities \( Ax - x_0 \mathbf{1} \leq b \) is

\[ a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \leq b_i + x_0. \]
The Auxiliary Problem

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\[ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i + x_0. \]

In block matrix form we write

\[
\begin{bmatrix}
-\mathbf{1} & A
\end{bmatrix}
\begin{pmatrix}
x_0 \\
x
\end{pmatrix}
\leq b.
\]
The Auxiliary Problem

\[ Q \quad \text{minimize} \quad x_0 \]

subject to

\[ Ax - x_0 1 \leq b, \quad 0 \leq x_0, x. \]
The Auxiliary Problem

\[ Q \quad \text{minimize} \quad x_0 \]
\[ \text{subject to} \quad Ax - x_0 1 \leq b, \quad 0 \leq x_0, x. \]

If the optimal value in the auxiliary problem is zero, then at the optimal solution \((\tilde{x}_0, \tilde{x})\) we have \(\tilde{x}_0 = 0\).
The Auxiliary Problem

\[ \begin{align*}
Q \quad & \text{minimize} \quad x_0 \\
\text{subject to} \quad & Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x.
\end{align*} \]

If the optimal value in the auxiliary problem is zero, then at the optimal solution \((\tilde{x}_0, \tilde{x})\) we have \(\tilde{x}_0 = 0\).

Plugging into \(Ax - x_0 \mathbf{1} \leq b\), we get \(A\tilde{x} \leq b\), i.e. \(\tilde{x}\) is feasible for \(\mathcal{P}\).
The Auxiliary Problem

\[ Q \begin{array}{l}
\text{minimize} \\
\text{subject to}
\end{array} x_0
\]
\[ Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x. \]

If the optimal value in the auxiliary problem is zero, then at the optimal solution \((\hat{x}_0, \hat{x})\) we have \(\hat{x}_0 = 0\).

Plugging into \(Ax - x_0 \mathbf{1} \leq b\), we get \(A\hat{x} \leq b\), i.e. \(\hat{x}\) is feasible for \(P\).

On the other hand, if \(\hat{x}\) is feasible for \(P\), then \((\hat{x}_0, \hat{x})\) with \(\hat{x}_0 = 0\) is feasible for \(Q\), so \((\hat{x}_0, \hat{x})\) is optimal for \(Q\).
The Auxiliary Problem

\[ \begin{align*}
\mathcal{P} & \quad \text{maximize} \quad c^T x \\
\text{subject to} \quad & \quad Ax \leq b, \\
& \quad 0 \leq x
\end{align*} \]

\[ \begin{align*}
\mathcal{Q} & \quad \text{minimize} \quad x_0 \\
\text{subject to} \quad & \quad Ax - x_0 \mathbf{1} \leq b, \\
& \quad 0 \leq x_0, x
\end{align*} \]

\( \mathcal{P} \) is feasible \( \iff \) the optimal value in \( \mathcal{Q} \) is zero.

\( \mathcal{P} \) is infeasible \( \iff \) the optimal value in \( \mathcal{Q} \) is positive.
The Auxiliary Problem

\[ \mathcal{P} \quad \text{maximize} \quad c^T x \quad \text{subject to} \quad Ax \leq b, \quad 0 \leq x \]

\[ Q \quad \text{minimize} \quad x_0 \quad \text{subject to} \quad Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x \]

- \( \mathcal{P} \) is feasible \( \iff \) the optimal value in \( Q \) is zero.
The Auxiliary Problem

\[ \mathcal{P} \quad \text{maximize} \quad c^T x \]
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- \( \mathcal{P} \) is feasible \( \iff \) the optimal value in \( \mathcal{Q} \) is zero.

- \( \mathcal{P} \) is infeasible \( \iff \) the optimal value in \( \mathcal{Q} \) is positive.
The auxiliary problem $Q$ is also called the *Phase I* problem since solving it is the first phase of a two phase process of solving general LPs.
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In Phase I we solve the auxiliary problem to obtain an initial feasible tableau for $P$, and in Phase II we solve the original LP starting with the feasible tableau provided in Phase I.
Initializing the Auxiliary Problem

\[
\begin{align*}
Q & \quad \text{minimize} & & x_0 \\
\text{subject to} & & & Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x.
\end{align*}
\]
Initializing the Auxiliary Problem

\[ Q \minimize x_0 \]
\[ \text{subject to } Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x. \]

Problem: The initial dictionary for \( Q \) is infeasible!
Initializing the Auxiliary Problem

\[ Q \text{ minimize } x_0 \]
\[ \text{subject to } Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x. \]

Problem: The initial dictionary for \( Q \) is infeasible!

Solution: Set \( x_0 = -\min\{b_i : i = 0, \ldots, n\} \) with \( b_0 = 0 \),
then \( b + x_0 \mathbf{1} \geq 0 \) since
\[
\min\{b_i + x_0 : i = 1, \ldots, m\} = \min\{b_i : i = 1, \ldots, m\} + x_0
\]
\[
= \min\{b_i : i = 1, \ldots, m\} - \min\{b_i : i = 0, \ldots, m\} \geq 0.
\]
Initializing the Auxiliary Problem

\[ \begin{align*}
Q & \quad \text{minimize} \quad x_0 \\
\text{subject to} & \quad Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x.
\end{align*} \]

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Solution: Set \( x_0 = -\min\{b_i : i = 0, \ldots, n\} \) with \( b_0 = 0 \), then \( b + x_0 \mathbf{1} \geq 0 \) since

\[ \begin{align*}
\min\{b_i + x_0 : i = 1, \ldots, m\} &= \min\{b_i : i = 1, \ldots, m\} + x_0 \\
&= \min\{b_i : i = 1, \ldots, m\} - \min\{b_i : i = 0, \ldots, m\} \geq 0.
\end{align*} \]

Hence, \( x_0 = -\min\{b_i : i = 0, \ldots, m\} \) and \( x = 0 \) is feasible for \( Q \).
Initializing the Auxiliary Problem

\[ \begin{align*}
Q \quad & \text{minimize} \quad x_0 \\
\text{subject to} \quad & Ax - x_0 1 \leq b, \quad 0 \leq x_0, x.
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\[ \min\{b_i + x_0 : i = 1, \ldots, m\} = \min\{b_i : i = 1, \ldots, m\} + x_0 \]
\[ = \min\{b_i : i = 1, \ldots, m\} - \min\{b_i : i = 0, \ldots, m\} \geq 0. \]

Hence, \( x_0 = -\min\{b_i : i = 0, \ldots, m\} \) and \( x = 0 \) is feasible for \( Q \).
It is also a BFS for \( Q \).
Initializing the Auxiliary Problem

\[ Q \minimize x_0 \text{ subject to } Ax - x_0 1 \leq b, \quad 0 \leq x_0, x. \]
Initializing the Auxiliary Problem

\[ \min \ x_0 \]
subject to \[ Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x. \]

The initial dictionary for \( Q \) is

\[
x_{n+i} = b_i + x_0 - \sum_{j=1}^{m} a_{ij} x_j
\]

\[
z = -x_0.
\]
Initializing the Auxiliary Problem

\[ Q \text{ minimize } x_0 \]
\[ \text{subject to } Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x. \]

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\[ x_{n+i} = b_i + x_0 - \sum_{j=1}^{m} a_{ij} x_j \]

\[ z = -x_0. \]

Let \( i_0 \) be such that \( b_{i_0} = \min\{b_i : i = 0, 1, \ldots, m\}. \)
Initializing the Auxiliary Problem

\[ \begin{align*}
Q \minimize x_0 \\
\text{subject to} \\
Ax - x_0 \mathbf{1} &\leq b, \\
0 &\leq x_0, x.
\end{align*} \]

The initial dictionary for \( Q \) is

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\[ z = -x_0. \]

Let \( i_0 \) be such that

\[ b_{i_0} = \min\{b_i : i = 0, 1, \ldots, m\}. \]

If \( i_0 = 0 \),
Initializing the Auxiliary Problem

\[ \min_{x_0} \quad x_0 \]
subject to \[ Ax - x_0 \mathbf{1} \leq b, \quad 0 \leq x_0, x. \]

The initial dictionary for \( Q \) is

\[ x_{n+i} = b_i + x_0 - \sum_{j=1}^{m} a_{ij}x_j \]

\[ z = -x_0. \]

Let \( i_0 \) be such that

\[ b_{i_0} = \min\{b_i : i = 0, 1, \ldots, m\}. \]

If \( i_0 = 0 \), the LP has feasible origin and so the initial dictionary is optimal.
Initializing the Auxiliary Problem

If $i_0 > 0$, then pivot on this row bringing $x_0$ into the basis yielding

\[
x_0 = -b_{i_0} + x_{n+i_0} + \sum_{j=1}^{m} a_{i_0j}x_j
\]

\[
x_{n+i} = b_i - b_{i_0} + x_{n+i_0} - \sum_{j=1}^{m} (a_{ij} - a_{i_0j})x_j, \quad i \neq i_0
\]

\[
z = b_{i_0} - x_{n+i_0} - \sum_{j=1}^{m} a_{i_0j}x_j.
\]
Initializing the Auxiliary Problem

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\]

\[
z = b_{i_0} - x_{n+i_0} - \sum_{j=1}^{m} a_{i_0j} x_j.
\]

This dictionary is feasible for \( Q \).
Initializing the Auxiliary Problem: Example

\[ \begin{align*}
\text{max} \quad & x_1 - x_2 + x_3 \\
\text{s.t.} \quad & 2x_1 - x_2 + 2x_3 \leq 4 \\
& 2x_1 - 3x_2 + x_3 \leq -5 \\
& -x_1 + x_2 - 2x_3 \leq -1 \\
& 0 \leq x_1, x_2, x_3.
\end{align*} \]
Initializing the Auxiliary Problem: Example

\[
\begin{align*}
\text{max} & \quad x_1 - x_2 + x_3 \\
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& \quad -x_1 + x_2 - 2x_3 \leq -1 \\
& \quad 0 \leq x_1, x_2, x_3.
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad -x_0 \\
\text{s.t.} & \quad -x_0 + 2x_1 - x_2 + 2x_3 \leq 4 \\
& \quad -x_0 + 2x_1 - 3x_2 + x_3 \leq -5 \\
& \quad -x_0 - x_1 + x_2 - 2x_3 \leq -1 \\
& \quad 0 \leq x_0, x_1, x_2, x_3.
\end{align*}
\]
Example

\[
\begin{align*}
\text{max} & \quad -x_0 \\
\text{s.t.} & \quad -x_0 + 2x_1 - x_2 + 2x_3 \leq 4 \\
& \quad -x_0 + 2x_1 - 3x_2 + x_3 \leq -5 \\
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Example

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& \quad -x_0 - x_1 + x_2 - 2x_3 \leq -1 \\
& \quad 0 \leq x_0, x_1, x_2, x_3.
\end{align*}
\]

\[
\begin{array}{cccccccc|c}
-1 & 2 & -1 & 2 & 1 & 0 & 0 & 4 \\
-1 & 2 & -3 & 1 & 0 & 1 & 0 & -5 \\
-1 & -1 & 1 & -2 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{array}
\]
### First Pivot

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First Pivot

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-5 most negative

\[ \begin{array}{cccccccc}
-1 & 2 & -1 & 2 & 1 & 0 & 0 & 4 \\
2 & -3 & 1 & 0 & 1 & 0 & -5 & 0 \\
-1 & -1 & 1 & -2 & 0 & 0 & 1 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]
### First Pivot

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<thead>
<tr>
<th>$x_0$</th>
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- **z**
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  - \(-2\) | 1 | 0 | 0 | 1 | 0 | 1 |

- **w**
  - 0 | \( \frac{1}{4} \) | 0 | \( \frac{5}{4} \) | 0 | \(-\frac{1}{4}\) | \(-\frac{3}{4}\) |
  - \(\frac{4}{5}\) | \(\frac{1}{5}\) | 0 | 1 | 0 | \(-\frac{1}{5}\) | \(-\frac{3}{5}\) |
  - \(\frac{3}{5}\) | \(-\frac{3}{5}\) | 1 | 0 | 0 | \(-\frac{2}{5}\) | \(-\frac{1}{5}\) |
  - \(-\frac{1}{5}\) | \(\frac{4}{20}\) | 0 | 0 | 0 | \(-\frac{4}{20}\) | \(\frac{8}{20}\) |
  - \(-1\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| 1 | 2 | 1 | 3 | 8 | 11 | 3 | 0 |

\[ \begin{array}{cccccccc}
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1 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
0 & -\frac{3}{4} & 1 & -\frac{3}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\midrule
z & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{4} & 1 \\
\midrule
w & 0 & \frac{1}{4} & 0 & \frac{5}{4} & 0 & -\frac{1}{4} & -\frac{3}{4} & 2 \\
\midrule
-2 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\midrule
\frac{4}{5} & \frac{1}{5} & 0 & 1 & 0 & -\frac{1}{5} & -\frac{3}{5} & 8 \\
\frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 11 \\
\midrule
z & -\frac{1}{5} & \frac{4}{20} & 0 & 0 & 0 & -\frac{4}{20} & \frac{8}{20} & 3 \\
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| –2  | 1   | 0   | 0   | 0   | 1   | 0    | 1    | 3  |
| 4/5 | 1/5 | 0   | 1   | 0   | -1/5| -3/5 | 8/5  |
| 3/5 | -3/5| 1   | 0   | 0   | -2/5| -1/5 | 11/5 |

| z   | -1/5| 4/20| 0   | 0   | 0   | -4/20| 8/20 | 3/5|
| w   | -1  | 0   | 0   | 0   | 0   | 0    | 0    | 0  |

Auxiliary problem solved.
### Auxiliary Problem Solution

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Original LP is feasible.
Extract Initial Feasible Tableau

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\frac{3}{5} & -\frac{3}{5} & 1 & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{11}{5} \\
\hline
z & -\frac{1}{5} & \frac{4}{20} & 0 & 0 & 0 & -\frac{4}{20} & \frac{8}{20} & \frac{3}{5} \\
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w & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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Extract initial.
feasible tableau.
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- Extract initial feasible tableau.

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<td>0</td>
<td>1</td>
<td>3/5</td>
<td>-1/5</td>
<td>0</td>
<td>17/5</td>
</tr>
<tr>
<td>-2/5</td>
<td>1</td>
<td>0</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>14/5</td>
</tr>
<tr>
<td>-1/5</td>
<td>0</td>
<td>0</td>
<td>-2/5</td>
<td>-1/5</td>
<td>0</td>
<td>-3/5</td>
</tr>
</tbody>
</table>
Phase II: Solution

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{bmatrix}
$$
Phase II: Solution

Optimal primal and dual solutions are

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{bmatrix}
\]
### Phase II: Solution

Optimal primal and dual solutions are

\[
\begin{align*}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} &=
\begin{bmatrix}
    0 \\
    2.8 \\
    3.4
\end{bmatrix}
\end{align*}
\]
### Phase II: Solution

\[
\begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 1 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 \\
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 \\
\end{array}
\]

\[
\begin{array}{c}
3 \\
\frac{17}{5} \\
\frac{14}{5} \\
-\frac{3}{5} \\
\end{array}
\]

Optimal primal and dual solutions are

\[
\begin{pmatrix}
\chi_1 \\
\chi_2 \\
\chi_3
\end{pmatrix} = 
\begin{pmatrix}
0 \\
2.8 \\
3.4
\end{pmatrix}
\]

\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3
\end{pmatrix} = 
\begin{pmatrix}
0.4 \\
0.2 \\
0
\end{pmatrix}
\]

Phase II: Solution

\[
\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 0 & 1 & 3 \\
\frac{4}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{17}{5} \\
-\frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & \frac{14}{5} \\
-\frac{1}{5} & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{3}{5}
\end{array}
\]

Optimal primal and dual solutions are

\[
\begin{pmatrix}
\chi_1 \\
\chi_2 \\
\chi_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
2.8 \\
3.4
\end{pmatrix}
\]

\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3
\end{pmatrix} =
\begin{pmatrix}
0.4 \\
0.2 \\
0
\end{pmatrix}
\]

with optimal value \( z = 0.6 \).
Steps for Phase I of the Two Phase Simplex Algorithm

We assume $b_{i_0} = \min\{b_i : i = 1, \ldots, m\} < 0$.

1. Form the standard initial tableau:
   \[
   \begin{bmatrix}
   0 & A & I & b \\
   -1 & c & 0 & 0
   \end{bmatrix}.
   \]

2. Border the initial tableau:
   \[
   \begin{bmatrix}
   -1 & 0 & A & I & b \\
   0 & -1 & c & 0 & 0 \\
   -1 & 0 & 0 & 0 & 0
   \end{bmatrix}.
   \]

3. In the first pivot, the pivot row is the $i_0$ row and the pivot column is the first column (the $x_0$ column).

4. Apply simplex algorithm on the $w$ row until optimality.

5. If optimal value is positive, stop the original LP is not feasible.

6. If the optimal value is zero, extract feasible tableau for the original problem and pivot to optimality.
Use the two phase simplex method to solve the following LP:

$$\text{maximize} \quad 3x_1 + x_2$$

$$\text{subject to} \quad x_1 - x_2 \leq -1$$

$$-x_1 - x_2 \leq -3$$

$$2x_1 + x_2 \leq 4$$

$$0 \leq x_1, x_2$$

Hint: A complete solution is possible in 3 pivots.