Linear Programming

Lecture 7: Does the Simplex Algorithm Work?

Math Dept, University of Washington

Does the Simples Algorithm Work?

Choosing Entering and Leaving Variables

Unbounded LPs

Degeneracy

Overcoming Degeneracy

Cycling

The Basis-Dictionary Correspondence

Bland's Rule for Pivoting

What Can Go Wrong with Simplex Algorithm?

Initialization:

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Hence, in order to pivot, we need an initial feasible dictionary.

How do we obtain the first feasible dictionary?

What Can Go Wrong with Simplex Algorithm?

Iteration:

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Iteration: Can we always choose variables to enter and leave the basis in an unambiguous way?

Can there be multiple choices or no choice?

Are there ambiguities in the choice of these variables, and these ambiguities be satisfactorily resolved?

What Can Go Wrong with Simplex Algorithm?

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Termination: Does the simplex algorithm terminate after a finite number of pivots?

Does it terminate at a solution when a solution exists?

Does it terminate when the problems is unbounded?

Can it stall, or can it go on pivoting forever without ever *solving* the problem?

Choosing the Entering Variable

Assume we are given a feasible dictionary:

$$\begin{aligned} x_i &= \widehat{b}_i - \sum_{j \in N} \widehat{a}_{ij} x_j \qquad\qquad i \in B \\ z &= \widehat{z} + \sum_{j \in N} \widehat{c}_j x_j \ , \end{aligned} \tag{D_B}$$

where $\widehat{b}_i \geq 0, \ i \in B$.

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A nonbasic variable x_{j_0}, $j_0 \in N$ can enter the basis if $\widehat{c}_{j_0} > 0$.

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Entering Variable:
A nonbasic variable x_{j_0}, $j_0 \in N$ can enter the basis if $\widehat{c}_{j_0} > 0$.

There may be many such nonbasic variables, but all of them have the potential to increase the value of the objective variable z.

Choosing the Leaving Variable

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The leaving variable attains the minimum value among the ratios

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If x_{i_0}, $i_0 \in B$ is the leaving variable, then

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\frac{\widehat{b}_{i_0}}{\widehat{a}_{i_0 j_0}} = \min \left\{ \frac{\widehat{b}_i}{\widehat{a}_{ij_0}} : i \in B, \ \widehat{a}_{ij_0} > 0 \right\}.
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(i) There is no $i \in B$ for which $\widehat{a}_{ij_0} > 0$.

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If (i) occurs, then we can increase the value of the entering variable x_{j_0}, as much as we want without violating feasibility. That is, we can increase the value of z as much as we want. Hence the LP is unbounded.

Unbounded LPs

Fact: If there exists $j_0 \in N$ in the dictionary D_B for which $\widehat{c}_{j_0} > 0$ and $\widehat{a}_{ij_0} \leq 0$ for all $i \in B$, then the LP

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b,\ 0 \leq x \end{array}$$

is unbounded, i.e., the optimal value is $+\infty$.

Unbounded LPs

$$
\begin{array}{lrrrrr}
\text{maximize} & x_1 & +x_2 & +x_3 & & \\
\text{subject to} & 3x_1 & +x_2 & -2x_3 & \leq & 5 \\
& 4x_1 & +3x_2 & & \leq & 7 \\
& 0 \leq x_1, x_2, x_3 & & & &
\end{array}
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\end{array}
$$

$$
\left[\begin{array}{cccccc|c}
3 & 1 & -2 & 1 & 0 & 0 & 5 \\
4 & 3 & 0 & 0 & 1 & 0 & 7 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}\right]
$$

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(ii) is **VERY BAD** for the simplex algorithm!

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(ii) is **VERY BAD** for the simplex algorithm!
We show why by example.

Example

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - x_2 + 8x_3 \\
\text{subject to} & 2x_1 - 4x_2 + 6x_3 \leq 3 \\
& -x_1 + 3x_2 + 4x_3 \leq 2 \\
& 2x_3 \leq 1 \\
& 0 \leq x_1, x_2, x_3
\end{array}
$$

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$$

$$
x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$z = 0$

2	−4	6	1	0	0	3
−1	3	4	0	1	0	2
0	0	②	0	0	1	1
2	−1	8	0	0	0	0

Note that any one of these rows could serve as the pivot row!

Example

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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Note that any one of these rows could serve as the pivot row!

$z = 0$

$$x = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

②	-4	0	1	0	-3	0
-1	3	0	0	1	-2	0
0	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
2	-1	0	0	0	-4	-4

Note that by pivoting on this tableau we do not change the objective value

$z = 4$

Example

$$x = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

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$$x = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

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1	-2	0	$\frac{1}{2}$	0	$-\frac{3}{2}$	0
0	①	0	$\frac{1}{2}$	1	$-\frac{7}{2}$	0
0	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
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Note that we have not changed the point identified by this tableau

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$$x = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

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1	0	0	$\frac{3}{2}$	2	$-\frac{17}{2}$	0
0	1	0	$\frac{1}{2}$	1	$-\frac{7}{2}$	0
0	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
0	0	0	$-\frac{5}{2}$	-3	$\frac{19}{2}$	-4

Again no change.

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0	0	1	0	0	$\textcircled{\frac{1}{2}}$	$\frac{1}{2}$
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Again no change.

$$x = \begin{pmatrix} \frac{17}{2} \\ \frac{7}{2} \\ 0 \end{pmatrix}$$

$$z = \frac{27}{2}$$

1	0	17	$\frac{3}{2}$	2	0	$\frac{17}{2}$
0	1	7	$\frac{1}{2}$	1	0	$\frac{7}{2}$
0	0	2	0	0	1	1
0	0	-19	$-\frac{5}{2}$	-3	0	$-\frac{27}{2}$

Finally, we break out to optimality.

Degeneracy

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Correspondingly, a tableau in which one or more of the basic variables is set to zero in the associated basic feasible solution is called a "degenerate tableau".

(2) It is possible that a pivot on a degenerate dictionary (or tableau) does not change the associated basic feasible solution and the value of the objective variable z.
Such a pivot is called a "degenerate pivot".

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Observation (2) is particularly troublesome since it opens the door to the possibility of an infinite sequence of degenerate pivots never terminating with optimality.

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Unfortunately, this *can* occur leading to the failure of the method. An example of the phenomenon is given in the text.

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Unfortunately, this *can* occur leading to the failure of the method. An example of the phenomenon is given in the text.

Our goal is to understand how such a pathological situation can occur and then to devise methods to overcome the problem.

Cycling

Assume the algorithm is operating with iron clad pivoting rules.

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- Choice of Entering Variable: Among all those variables x_j with $j \in N$ such that $\widehat{c}_j = \max\{\widehat{c}_k : k \in N\} > 0$ choose x_{j_0} so that j_0 is largest.

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- Choice of Leaving Variable: Among all those variables x_i with $i \in B$ such that $\frac{\widehat{b}_i}{\widehat{a}_{ij_0}} = \min\left\{\frac{\widehat{b}_k}{\widehat{a}_{kj_0}} : k \in B, \widehat{a}_{kj_0} > 0\right\}$ choose x_{i_0} so that i_0 is largest.

How Many Possible Bases?

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Since every basis must contain m variables and there are only $n + m$ variables altogether, the total number of possible sets of basic indices equals the number of possible ways to choose m distinct elements from a collection of $n + m$ objects.

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$$\left(\begin{array}{c} n+m \\ m \end{array} \right) = \frac{(n+m)!}{m!n!}.$$

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Suppose B is also the basis for D_{N+1}.
Let

$$\ldots D_1, \ldots, D_N, D_{N+1}, D_{N+2}, D_{N+2}, \ldots$$

be the sequence of pivots where D_1 and D_{N+1} have the same basis.

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If each basis is associated with a unique dictionary, then
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be the sequence of pivots where D_1 and D_{N+1} have the same basis.
If each basis is associated with a unique dictionary, then
$D_1 = D_{N+1}$. But then,

$$D_2 = D_{N+2},\ D_3 = D_{N+3},\ \ldots,\ D_1 = D_{2N},\ D_2 = D_{2N+2},\ \ldots$$

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Suppose B is also the basis for D_{N+1}.
Let

$$\ldots D_1, \ldots, D_N, D_{N+1}, D_{N+2}, D_{N+2}, \ldots$$

be the sequence of pivots where D_1 and D_{N+1} have the same basis.
If each basis is associated with a unique dictionary, then
$D_1 = D_{N+1}$. But then,

$$D_2 = D_{N+2},\ D_3 = D_{N+3},\ \ldots,\ D_1 = D_{2N},\ D_2 = D_{2N+2},\ \ldots$$

That is, the same sequence of dictionaries appear over and over again. If this occurs we say that the sequence of dictionaries cycles.

The Basis-Dictionary Correspondence

Fact:
The simplex algorithm fails to terminate if and only if it cycles. The simplex algorithms can only cycle between degenerate dictionaries (or tableaus) with each dictionary (or tableau) in the cycle being associated with the same basic feasible solution and objective value.

Proof

We show each basis yields a unique dictionary.

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$$\begin{aligned} x_i &= \widehat{b}_i - \sum_{j \in N} \widehat{a}_{ij} x_j, \qquad i \in B \\ z &= \widehat{z}_i + \sum_{j \in N} \widehat{c}_j x_j \end{aligned} \tag{D_1}$$

and

$$\begin{aligned} x_i &= b_i^* - \sum_{j \in N} a_{ij}^* x_j, \qquad i \in B \\ z &= z^* + \sum_{j \in N} c_j^* x_j \end{aligned} \tag{D_2}$$

Two dictionaries with the same basis B.

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Two dictionaries with the same basis B.
Show all coefficients are identical.

Proof

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\begin{array}{rcl}
 & (D_1) & \\
x_i & = & \widehat{b}_i - \sum_{j \in N} \widehat{a}_{ij} x_j, \quad i \in B \\
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\end{array}
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D_1 and D_2 have identical solution sets.

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D_1 and D_2 have identical solution sets.
Let $j_0 \in N$ and set $x_{j_0} = t$ and $x_j = 0$ for $j \in N$, $j \neq j_0$.
Then

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\begin{array}{rcll}
\widehat{b}_i - \widehat{a}_{ij_0} t & = & x_i = b_i^* - a_{ij_0}^* t & \text{for } i \in B \\
\widehat{z} + \widehat{c}_{j_0} t & = & z = z^* + c_{j_0}^* t. &
\end{array}
$$

Setting $t = 0$, we have

$$
\widehat{b}_i = b_i^* \quad i \in B \quad \text{and} \quad \widehat{z} = z^*.
$$

Then, setting $t = 1$, we have

$$
\widehat{a}_{ij_0} = a_{ij_0}^* \qquad \text{for } i \in B.
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Repeating for all $j \in N$.

Degeneracy and Cycling

We have established that the simplex algorithm can only fail to terminate if it cycles, and that it can only cycle in the presence of degeneracy. In order to assure that the simplex algorithm successfully terminates we need to develop a pivoting rule that avoids cycling.

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There are many anti-cycling pivoting rules. We present the smallest subscript rule, also known as Bland's Rule.

Bland's Rule

$$\begin{aligned} x_i &= \widehat{b}_i - \sum_{j \in N} \widehat{a}_{ij} x_j \qquad i \in B \\ z &= \widehat{z} + \sum_{j \in N} \widehat{c}_j x_j \ . \end{aligned} \tag{D_B}$$

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Choice of entering variable: x_{j_0} for $j_0 \in N$ is the entering variable if $\widehat{c}_{j_0} > 0$ and $j_0 \leq j$ whenever $\widehat{c}_j > 0$.

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Choice of entering variable: x_{j_0} for $j_0 \in N$ is the entering variable if $\widehat{c}_{j_0} > 0$ and $j_0 \leq j$ whenever $\widehat{c}_j > 0$.

Choice of leaving variable: x_{i_0} for $i_0 \in B$ is the leaving variable if

$$\frac{\widehat{b}_{i_0}}{\widehat{a}_{i_0 j_0}} = \min \left\{ \frac{\widehat{b}_i}{\widehat{a}_{i j_0}} : i \in B, \widehat{a}_{i j_0} > 0 \right\}$$

and

$$i_0 \leq i \ \text{ whenever } \ \frac{\widehat{b}_{i_0}}{\widehat{a}_{i_0 j_0}} = \frac{\widehat{b}_i}{\widehat{a}_{i j_0}} \quad i \in B.$$

Bland's Rule

Theorem: [R.G. Bland (1977)]
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We abbreviate the *Bland's rule* to **SSR** (smallest subscript rule).

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Prove Bland's rule prevents cycling.

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Denote the set of fickle variables by $\mathcal{F}$.

All fickle variables must take the value zero in the BFS associated with this cycle since they take the value zero when they are not in the basis.

Proof

Let ℓ be the largest subscript in $\mathcal{F}$, and let

$$D \qquad \begin{aligned} x_i &= b_i - \sum_{j \notin B} a_{ij} x_j \,, \qquad i \in B \\ z &= v + \sum_{j \notin B} c_j x_j \end{aligned}$$

be a dictionary in the cycle where x_ℓ is leaving the basis and let x_e denote the entering variable: x_ℓ leave D and x_e enters.

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Let D^* be a dictionary in the cycle with x_ℓ entering the basis. Since each dictionary in the cycle identifies the same BFS, the objective value v stays constant throughout the cycle.

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Write the objective row of D^* (x_ℓ entering) as

$$z = v + \sum_{j=1}^{m+n} c_j^* x_j,$$

where $c_j^* = 0$ if x_j is basic in D^*. Note, $c_j^* \leq 0 \; \forall \; j \in \mathcal{F} \setminus \{\ell\}$.

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$$x_i = b_i - a_{ie} t \; (i \in B) \text{ and } z = v + c_e t$$

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Hence $\quad v + c_e t = v + c_e^* t + \sum_{i \in B} c_i^* (b_i - a_{ie} t) \qquad$ for all t.

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Grouping terms gives

$$\left(c_e - c_e^* + \sum_{i \in B} c_i^* a_{ie} \right) t = \sum_{i \in B} c_i^* b_i \qquad \text{for all } t.$$

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Since the right hand side is constant, it must be zero as is the coefficient on the left:

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Therefore, $c_e - c_e^* > 0$.

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Therefore, $c_e - c_e^* > 0$.

Consequently, $\sum_{i \in B} c_i^* a_{ie} < 0$, so for some $s \in B$,

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We claim that $s < \ell$.

Proof

$$s \in B \quad c_s^* a_{se} < 0 \quad x_s \text{ not basic in } D^*$$

Show $s < \ell$.

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Consequently, $c_\ell^* a_{\ell e} > 0$, so $s < \ell$.

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Since $s < \ell$, x_s cannot be a candidate to enter the basis in D^* by **SSR**, that is, $c_s^* < 0$.

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But then $a_{se} > 0$, since $c_s^* a_{se} < 0$.

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$x_s \in \mathcal{F}$ and $s \in B$, so the value of x_s in the BFS is zero, $(b_s = 0)$.

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Since $b_s = 0$ and $a_{se} > 0$, x_s is a candidate for leaving in D.

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This contradicts the **SSR**, so no cycle can exist.