Linear Programming

Lecture 7: Does the Simplex Algorithm Work?

Math Dept, University of Washington

What Can Go Wrong with Simplex Algorithm?

Initialization:

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Hence, in order to pivot, we need an initial feasible dictionary.

How do we obtain the first feasible dictionary?

What Can Go Wrong with Simplex Algorithm?

Iteration:

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Iteration: Can we always choose variables to enter and leave the basis in an unambiguous way?

Can there be multiple choices or no choice?

Are there ambiguities in the choice of these variables, and these ambiguities be satisfactorily resolved?

What Can Go Wrong with Simplex Algorithm?

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Termination: Does the simplex algorithm terminate after a finite number of pivots?

Does it terminate at a solution when a solution exists?

Does it terminate when the problems is unbounded?

Can it stall, or can it go on pivoting forever without ever *solving* the problem?

Choosing the Entering Variable

Assume we are given a feasible dictionary:

$$\begin{array}{rcll} x_i & = & \widehat{b}_i - \sum_{j\in N} \widehat{a}_{ij} x_j & \quad i \in B \\ z & = & \widehat{z} + \sum_{j\in N} \widehat{c}_j x_j \ , & \end{array} \qquad (D_B)$$

where $\widehat{b}_i \geq 0, \ i \in B$.

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Entering Variable:
A nonbasic variable x_{j_0}, $j_0 \in N$ can enter the basis if $\widehat{c}_{j_0} > 0$.

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Entering Variable:
A nonbasic variable x_{j_0}, $j_0 \in N$ can enter the basis if $\widehat{c}_{j_0} > 0$.

There may be many such nonbasic variables, but all of them have the potential to increase the value of the objective variable z.

Choosing the Leaving Variable

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Suppose x_{j_0}, $j_0 \in N$ is the entering variable.

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The leaving variable attains the minimum value among the ratios

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If x_{i_0}, $i_0 \in B$ is the leaving variable, then

$$\frac{\widehat{b}_{i_0}}{\widehat{a}_{i_0 j_0}} = \min \left\{ \frac{\widehat{b}_i}{\widehat{a}_{ij_0}} : i \in B, \ \widehat{a}_{ij_0} > 0 \right\}.$$

Hazards in Choosing the Leaving Variable

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(i) There is no $i \in B$ for which $\widehat{a}_{ij_0} > 0$.

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If (i) occurs, then we can increase the value of the entering variable x_{j_0}, as much as we want without violating feasibility.

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That is, we can increase the value of z as much as we want.
Hence the LP is unbounded.

Unbounded LPs

Fact: If there exists $j_0 \in N$ in the dictionary D_B for which $\widehat{c}_{j_0} > 0$ and $\widehat{a}_{ij_0} \leq 0$ for all $i \in B$, then the LP

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b,\ 0 \leq x \end{array}$$

is unbounded, i.e., the optimal value is $+\infty$.

Unbounded LPs

$$
\begin{array}{lrrrrr}
\text{maximize} & x_1 & +x_2 & +x_3 & & \\
\text{subject to} & 3x_1 & +x_2 & -2x_3 & \leq & 5 \\
 & 4x_1 & +3x_2 & & \leq & 7 \\
 & \multicolumn{5}{l}{0 \leq x_1, x_2, x_3}
\end{array}
$$

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$$\begin{array}{lrrrcl} \text{maximize} & x_1 & +x_2 & +x_3 & & \\ \text{subject to} & 3x_1 & +x_2 & -2x_3 & \leq & 5 \\ & 4x_1 & +3x_2 & & \leq & 7 \\ & \multicolumn{3}{l}{0 \leq x_1, x_2, x_3} & & \end{array}$$

$$\left[\begin{array}{cccccc|c} 3 & 1 & -2 & 1 & 0 & 0 & 5 \\ 4 & 3 & 0 & 0 & 1 & 0 & 7 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

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(ii) is **VERY BAD** for the simplex algorithm!

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(ii) is **VERY BAD** for the simplex algorithm!
We show why by example.

Example

$$\begin{array}{lr} \text{maximize} & 2x_1 - x_2 + 8x_3 \\ \text{subject to} & 2x_1 - 4x_2 + 6x_3 \leq 3 \\ & -x_1 + 3x_2 + 4x_3 \leq 2 \\ & 2x_3 \leq 1 \\ & 0 \leq x_1, x_2, x_3 \end{array}$$

Example

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\text{maximize} & 2x_1 - x_2 + 8x_3 \\
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 & 2x_3 \leq 1 \\
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\end{array}
$$

$$
x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad
\begin{array}{|cccccc|c}
2 & -4 & 6 & 1 & 0 & 0 & 3 \\
-1 & 3 & 4 & 0 & 1 & 0 & 2 \\
0 & 0 & 2 & 0 & 0 & 1 & 1 \\
\hline
2 & -1 & 8 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

$z = 0$

Note that any one of these rows could serve as the pivot row!

Example

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad z = 0$$

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$$x = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \qquad z = 4$$

2	-4	0	1	0	-3	0
-1	3	0	0	1	-2	0
0	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
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Note that by pivoting on this tableau we do not change the objective value

Example

$$x = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \qquad \begin{array}{|cccccc|c} 2 & -4 & 0 & 1 & 0 & -3 & 0 \\ -1 & 3 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline 2 & -1 & 0 & 0 & 0 & -4 & -4 \end{array}$$

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$$x = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \qquad \begin{array}{|cccccc|c} 1 & -2 & 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{7}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline 0 & 3 & 0 & -1 & 0 & -1 & -4 \end{array}$$

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$$x = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \qquad \begin{array}{cccccc|c} 1 & 0 & 0 & \frac{3}{2} & 2 & -\frac{17}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{7}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline 0 & 0 & 0 & -\frac{5}{2} & -3 & \frac{19}{2} & -4 \end{array}$$

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Again no change.

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$$x = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \qquad \begin{array}{|cccccc|c} 1 & 0 & 0 & \frac{3}{2} & 2 & -\frac{17}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{7}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline 0 & 0 & 0 & -\frac{5}{2} & -3 & \frac{19}{2} & -4 \end{array}$$

$z = 4$

Again no change.

$$x = \begin{pmatrix} \frac{17}{2} \\ \frac{7}{2} \\ 0 \end{pmatrix} \qquad \begin{array}{|cccccc|c} 1 & 0 & 17 & \frac{3}{2} & 2 & 0 & \frac{17}{2} \\ 0 & 1 & 7 & \frac{1}{2} & 1 & 0 & \frac{7}{2} \\ 0 & 0 & 2 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & \text{-}19 & -\frac{5}{2} & -3 & 0 & -\frac{27}{2} \end{array}$$

$z = \frac{27}{2}$

Finally, we break out to optimality.

Degeneracy

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Correspondingly, a tableau in which one or more of the basic variables is set to zero in the associated basic feasible solution is called a "degenerate tableau".

(2) It is possible that a pivot on a degenerate dictionary (or tableau) does not change the associated basic feasible solution and the value of the objective variable z.
Such a pivot is called a "degenerate pivot".

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Observation (2) is particularly troublesome since it opens the door to the possibility of an infinite sequence of degenerate pivots never terminating with optimality.

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Unfortunately, this *can* occur leading to the failure of the method. An example of the phenomenon is given in the text.

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Observation (2) is particularly troublesome since it opens the door to the possibility of an infinite sequence of degenerate pivots never terminating with optimality.

Unfortunately, this *can* occur leading to the failure of the method. An example of the phenomenon is given in the text.

Our goal is to understand how such a pathological situation can occur and then to devise methods to overcome the problem.

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- Choice of Entering Variable: Among all those variables x_j with $j \in N$ such that $\widehat{c}_j = \max\{\widehat{c}_k : k \in N\} > 0$ choose x_{j_0} so that j_0 is largest.

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How Many Possible Bases?

How many possible ways are there to choose a set of basic indices?

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Since every basis must contain m variables and there are only $n + m$ variables altogether, the total number of possible sets of basic indices equals the number of possible ways to choose m distinct elements from a collection of $n + m$ objects.

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$$\begin{pmatrix} n+m \\ m \end{pmatrix} = \frac{(n+m)!}{m!n!}.$$

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Let

$$\ldots D_1, \ldots, D_N, D_{N+1}, D_{N+2}, D_{N+2}, \ldots$$

be the sequence of pivots where D_1 and D_{N+1} have the same basis.

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If each basis is associated with a unique dictionary, then $D_1 = D_{N+1}$.

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be the sequence of pivots where D_1 and D_{N+1} have the same basis.
If each basis is associated with a unique dictionary, then $D_1 = D_{N+1}$. But then,

$$D_2 = D_{N+2},\ D_3 = D_{N+3},\ \ldots,\ D_1 = D_{2N},\ D_2 = D_{2N+2},\ \ldots$$

Cycling

$$\text{infinite pivot sequence} \Rightarrow \text{infinite dictionary sequence.}$$

At least one dictionary, say D_1, has a basis B appearing twice.
Suppose B is also the basis for D_{N+1}.
Let

$$\ldots D_1, \ldots, D_N, D_{N+1}, D_{N+2}, D_{N+2}, \ldots$$

be the sequence of pivots where D_1 and D_{N+1} have the same basis.
If each basis is associated with a unique dictionary, then $D_1 = D_{N+1}$. But then,

$$D_2 = D_{N+2},\ D_3 = D_{N+3},\ \ldots,\ D_1 = D_{2N},\ D_2 = D_{2N+2},\ \ldots$$

That is, the same sequence of dictionaries appear over and over again. If this occurs we say that the sequence of dictionaries cycles.

The Basis-Dictionary Correspondence

Fact:
The simplex algorithm fails to terminate if and only if it cycles. The simplex algorithms can only cycle between degenerate dictionaries (or tableaus) with each dictionary (or tableau) in the cycle being associated with the same basic feasible solution and objective value.

Proof

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$$\begin{array}{rcll} x_i & = & \widehat{b}_i - \sum_{j \in N} \widehat{a}_{ij} x_j, & \qquad i \in B \\ z & = & \widehat{z}_i + \sum_{j \in N} \widehat{c}_j x_j & \end{array} \qquad (D_1)$$

and

$$\begin{array}{rcll} x_i & = & b^*_i - \sum_{j \in N} a^*_{ij} x_j, & \qquad i \in B \\ z & = & z^* + \sum_{j \in N} c^*_j x_j & \end{array} \qquad (D_2)$$

Two dictionaries with the same basis B.

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$$\begin{array}{rcll} x_i & = & b_i^* - \sum_{j \in N} a_{ij}^* x_j, & \qquad i \in B \\ z & = & z^* + \sum_{j \in N} c_j^* x_j & \end{array} \qquad (D_2)$$

Two dictionaries with the same basis B.
Show all coefficients are identical.

Proof

$$
\begin{array}{rcl}
 & (D_1) & \\
x_i & = & \widehat{b}_i - \sum_{j\in N} \widehat{a}_{ij} x_j, \quad i \in B \\
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\end{array}
\qquad
\begin{array}{rcl}
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$$

D_1 and D_2 have identical solution sets.

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D_1 and D_2 have identical solution sets.
Let $j_0 \in N$ and set $x_{j_0} = t$ and $x_j = 0$ for $j \in N$, $j \neq j_0$.
Then

$$
\begin{array}{rcll}
\widehat{b}_i - \widehat{a}_{ij_0} t & = & x_i = b^*_i - a^*_{ij_0} t & \text{for } i \in B \\
\widehat{z} + \widehat{c}_{j_0} t & = & z = z^* + c^*_{j_0} t. &
\end{array}
$$

Setting $t = 0$, we have

$$\widehat{b}_i = b^*_i \quad i \in B \quad \text{and} \quad \widehat{z} = z^*.$$

Then, setting $t = 1$, we have

$$\widehat{a}_{ij_0} = a^*_{ij_0} \qquad \text{for } i \in B.$$

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Repeating for all $j \in N$.

Degeneracy and Cycling

We have established that the simplex algorithm can only fail to terminate if it cycles, and that it can only cycle in the presence of degeneracy. In order to assure that the simplex algorithm successfully terminates we need to develop a pivoting rule that avoids cycling.

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There are many anti-cycling pivoting rules. We present the smallest subscript rule, also known as Bland's Rule.

Bland's Rule

$$\begin{array}{rcll} x_i &=& \widehat{b}_i - \sum_{j\in N} \widehat{a}_{ij} x_j & \qquad i \in B \\ z &=& \widehat{z} + \sum_{j\in N} \widehat{c}_j x_j \ . & \end{array} \qquad (D_B)$$

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Choice of entering variable: x_{j_0} for $j_0 \in N$ is the entering variable if $\widehat{c}_{j_0} > 0$ and $j_0 \leq j$ whenever $\widehat{c}_j > 0$.

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Choice of leaving variable: x_{i_0} for $i_0 \in B$ is the leaving variable if

$$\frac{\widehat{b}_{i_0}}{\widehat{a}_{i_0 j_0}} = \min\left\{ \frac{\widehat{b}_i}{\widehat{a}_{ij_0}} : i \in B, \widehat{a}_{ij_0} > 0 \right\}$$

and

$$i_0 \leq i \quad \text{whenever} \quad \frac{\widehat{b}_{i_0}}{\widehat{a}_{i_0 j_0}} = \frac{\widehat{b}_i}{\widehat{a}_{ij_0}} \quad i \in B.$$

Bland's Rule

Theorem: [R.G. Bland (1977)]
The simplex algorithm terminates as long as the choice of variable to enter or leave the basis is made according to the smallest subscript rule.

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We abbreviate the *Bland's rule* to **SSR** (smallest subscript rule).

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We call the variables that leave the basis in any of the dictionaries $D_0, \ldots, D_N$ *fickle* since they move in and out of the basis.

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All fickle variables must take the value zero in the BFS associated with this cycle since they take the value zero when they are not in the basis.

Proof

Let ℓ be the largest subscript in $\mathcal{F}$, and let

$$D \qquad \begin{aligned} x_i &= b_i - \sum_{j \notin B} a_{ij} x_j\,, \qquad i \in B \\ z &= v + \sum_{j \notin B} c_j x_j \end{aligned}$$

be a dictionary in the cycle where x_ℓ is leaving the basis and let x_e denote the entering variable: x_ℓ leaves D and x_e enters.

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Since x_e is also fickle, $e < \ell$ (ℓ is largest in $\mathcal{F}$).

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Write the objective row of D^* (x_ℓ entering) as

$$z = v + \sum_{j=1}^{m+n} c_j^* x_j,$$

where $c_j^* = 0$ if x_j is basic in D^*. Note, $c_j^* \leq 0 \ \ \forall\, j \in \mathcal{F} \setminus \{\ell\}$.

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Since the solution sets to D and D^* coincide, the solution to D obtained by setting $x_e = t,\; x_j = 0$ ($j \notin B,\; j \neq e$) giving

$$x_i = b_i - a_{ie} t \;\; (i \in B) \text{ and } z = v + c_e t$$

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Hence $\quad v + c_e t = v + c_e^* t + \sum_{i \in B} c_i^* (b_i - a_{ie} t) \qquad$ for all t.

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Grouping terms gives

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Since the right hand side is constant, it must be zero as is the coefficient on the left:

$$c_e - c_e^* = -\sum_{i \in B} c_i^* a_{ie} \; .$$

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Therefore, $c_e - c_e^* > 0$.

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Therefore, $c_e - c_e^* > 0$.

Consequently, $\sum_{i \in B} c_i^* a_{ie} < 0$, so for some $s \in B$,

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Since $s \in B$, x_s is basic in D, and since $c_s^* \neq 0$, x_s is nonbasic in D^*, so $x_s \in \mathcal{F}$ which implies that $s \leq \ell$ (since ℓ is largest).

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We claim that $s < \ell$.

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$$s \in B \quad c_s^* a_{se} < 0 \quad x_s \text{ not basic in } D^*$$

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Consequently, $c_\ell^* a_{\ell e} > 0$, so $s < \ell$.

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$x_s \in \mathcal{F}$ and $s \in B$, so the value of x_s in the BFS is zero, $(b_s = 0)$.

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But x_ℓ leaves in D with $s < \ell$.

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This contradicts the **SSR**, so no cycle can exist.