Linear Programming

Lecture 6: The Simplex Algorithm
Language, Notation, and Linear Algebra

Math Dept, University of Washington

Dictionaries for LPs in Standard Form

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\mathcal{P}: & \text{maximize} & c^T x \\
 & \text{subject to} & Ax \leq b \\
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General Dictionaries

A dictionary for $\mathcal{P}$ is any system of the form

$$x_i = \widehat{b}_i - \sum_{j \in N} \widehat{a}_{ij} x_j \qquad i \in B \qquad (D_B)$$

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and such that the systems (D_I) and (D_B) have identical solution sets.

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- A feasible dictionary is *optimal* if $\widehat{c}_j \leq 0$ $j \in N$.

The Simplex Algorithm via Matrix Multiplication

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First recall that

$$\begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix} \begin{pmatrix} a \\ \alpha \\ b \end{pmatrix} = \begin{bmatrix} a-a \\ \alpha^{-1}\alpha \\ -b+b \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

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The elimination matrix and its inverse.

$$G = \begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix} \qquad G^{-1} = \begin{bmatrix} I & a & 0 \\ 0 & \alpha & 0 \\ 0 & b & I \end{bmatrix}$$

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Each simplex pivot can be realized as left multiplication of the simplex tableau by the appropriate Gaussian-Jordan pivot matrix.

The Simplex Algorithm via Matrix Multiplication

$$\left[\begin{array}{cccccc|c} 1 & 4 & 2 & 1 & 0 & 0 & 11 \\ 3 & 2 & 1 & 0 & 1 & 0 & 5 \\ 4 & 2 & 2 & 0 & 0 & 1 & 8 \\ \hline 4 & 5 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & \frac{-5}{2} & 0 & 1 \end{bmatrix} \left[\begin{array}{cccccc|c} 1 & 4 & 2 & 1 & 0 & 0 & 11 \\ 3 & 2 & 1 & 0 & 1 & 0 & 5 \\ 4 & 2 & 2 & 0 & 0 & 1 & 8 \\ \hline 4 & 5 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

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The Block Structure of the Simplex Algorithm

Let T_0 be the initial tableau:

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Let T_k denote the tableau after k pivots:

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T_k is obtained from T_0 by multiplying it on the left by a product of Gaussian pivot matrices $G := G_k G_{k-1} \cdots G_1$:

$$GT_0 = T_k,$$

where G is invertible ($G^{-1} = G_1^{-1} G_2^{-1} \cdots G_k^{-1}$).

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Let's investigate the structure of T_k by examining the consequence of this product in terms of the block structure of T_0 and T_k.

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First we must decompose G into a block structure that is conformal to that of T_0:

$$G = \left[\begin{array}{cc} M & u \\ v^{\mathsf{T}} & \beta \end{array}\right],$$

where $M \in \mathbb{R}^{m\times m}$, $u, v \in \mathbb{R}^m$, and $\beta \in \mathbb{R}$.

Block Structure and Matrix Multiplication

$$\left[\begin{array}{ccc|c} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^{\mathsf{T}} & -y^{\mathsf{T}} & \widehat{z} \end{array}\right] \quad = \quad T_k$$

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Equating terms on the left and right gives

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$$u = 0$$

Block Structure and Matrix Multiplication

$$\left[\begin{array}{rcr|c} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^{\mathsf{T}} & -y^{\mathsf{T}} & \widehat{z} \end{array}\right] = \left[\begin{array}{rcc|c} -u & MA + uc^{\mathsf{T}} & M & Mb \\ -\beta & v^{\mathsf{T}}A + \beta c^{\mathsf{T}} & v^{\mathsf{T}} & v^{\mathsf{T}}b \end{array}\right]$$

Equating terms on the left and right gives

$$u = 0 \qquad\qquad \beta = 1$$

Block Structure and Matrix Multiplication

$$\left[\begin{array}{ccc|c} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^{\mathsf{T}} & -y^{\mathsf{T}} & \widehat{z} \end{array}\right] = \left[\begin{array}{ccc|c} -u & MA + uc^{\mathsf{T}} & M & Mb \\ -\beta & v^{\mathsf{T}}A + \beta c^{\mathsf{T}} & v^{\mathsf{T}} & v^{\mathsf{T}}b \end{array}\right]$$

Equating terms on the left and right gives

$$\begin{aligned} u &= 0 & \qquad \beta &= 1 \\ M &= R \end{aligned}$$

Block Structure and Matrix Multiplication

$$\left[\begin{array}{ccc|c} 0 & \widehat{A} & R & \widehat{b} \\ -1 & \widehat{c}^\top & -y^\top & \widehat{z} \end{array}\right] = \left[\begin{array}{ccc|c} -u & MA + uc^\top & M & Mb \\ -\beta & v^\top A + \beta c^\top & v^\top & v^\top b \end{array}\right]$$

Equating terms on the left and right gives

$$\begin{array}{rcl} u = 0 & & \beta = 1 \\ M = R & \text{and} & v = -y. \end{array}$$

Block Structure and Matrix Multiplication

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Therefore,

$$T_k = \left[\begin{array}{cc} R & 0 \\ -y^{\mathsf{T}} & 1 \end{array}\right] \left[\begin{array}{ccc|c} 0 & A & I & b \\ -1 & c^{\mathsf{T}} & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^{\mathsf{T}} - y^{\mathsf{T}}A & -y^{\mathsf{T}} & -y^{\mathsf{T}}b \end{array}\right],$$

where the matrix R is necessarily invertible.

Block Structure and Matrix Multiplication

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where the matrix R is necessarily invertible. (R $\sim$record matrix)

The Block Structure of an Optimal Tableau

$$T_k = \left[\begin{array}{cc} R & 0 \\ -y^\top & 1 \end{array}\right] \left[\begin{array}{ccc|c} 0 & A & I & b \\ -1 & c^\top & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^\top - y^\top A & -y^\top & -y^\top b \end{array}\right]$$

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T_k is an optimal tableau if and only if

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T_k is an optimal tableau if and only if

- it is feasible: $0 \leq Rb$, and
- the z-row has non-positive entries:

$$c - A^\top y \leq 0 \quad \text{or equivalently} \quad A^\top y \geq c$$
$$-y \leq 0 \quad \text{or equivalently} \quad 0 \leq y.$$

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In this case the optimal value $= z = b^\top y$.

The Block Structure of an Optimal Tableau

$$\left[\begin{array}{cccc|c} 0 & RA & R & Rb \\ -1 & c^\top - y^\top A & -y^\top & -y^\top b \end{array}\right]$$

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$$\begin{array}{lll} \mathcal{P} & \max & c^\top x \\ & \text{s.t.} & Ax \le b \\ & & 0 \le x \end{array}$$

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The Block Structure of an Optimal Tableau

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WEAK DUALITY THM. $\Rightarrow$ Y SOLVES $\mathcal{D}$!!!

Optimal Tableaus Yield Optimal Solutions

$$\begin{array}{llll} \mathcal{P} & \max & c^{\mathsf{T}}x \\ & \text{s.t.} & Ax \le b \\ & & 0 \le x \end{array} \qquad \begin{array}{llll} \mathcal{D} & \min & b^{\mathsf{T}}y \\ & \text{s.t.} & A^{\mathsf{T}}y \ge c \\ & & 0 \le y \end{array}$$

Theorem: If the simplex tableau

$$\left[\begin{array}{ccc|c} 0 & RA & R & Rb \\ -1 & c^{\mathsf{T}} - y^{\mathsf{T}}A & -y^{\mathsf{T}} & -y^{\mathsf{T}}b \end{array}\right]$$

is optimal for $\mathcal{P}$, i.e. if x^* is the associated BFS and

$$0 \le Rb, \quad A^{\mathsf{T}}y \ge c, \quad 0 \le y, \quad c^{\mathsf{T}}x^* = z = b^{\mathsf{T}}y,$$

then x^* is an optimal solution to $\mathcal{P}$ and y is an optimal solution to $\mathcal{D}$.

Plastic Cup Factory Reprised

$$
\begin{array}{lll}
\mathcal{P} & \text{maximize} & 25B + 20C \\
 & \text{subject to} & 20B + 12C \leq 1800 \\
 & & \frac{1}{15}B + \frac{1}{15}C \leq 8 \\
 & & 0 \leq B, C
\end{array}
$$

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$$\begin{array}{lll} \mathcal{D} & \text{minimize} & 1800R + 8L \\ & \text{subject to} & 20R + \frac{1}{15}L \geq 25 \\ & & 12R + \frac{1}{15}L \geq 20 \\ & & 0 \leq R, L \end{array}$$

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$$\left[\begin{array}{cccc|c} 20 & 12 & 1 & 0 & 1800 \\ \frac{1}{15} & \frac{1}{15} & 0 & 1 & 8 \\ \hline 25 & 20 & 0 & 0 & 0 \end{array}\right]$$

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$$\left[\begin{array}{cccc|c} 20 & 12 & 1 & 0 & 1800 \\ \frac{1}{15} & \frac{1}{15} & 0 & 1 & 8 \\ \hline 25 & 20 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1/8 & -75/2 & 45 \\ 0 & 1 & -1/8 & 75/2 & 75 \\ \hline 0 & 0 & -5/8 & -375/2 & -2625 \end{array}\right]$$

Plastic Cup Factory Reprised

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & 25B + 20C \\ & \text{subject to} & 20B + 12C \le 1800 \\ & & \frac{1}{15}B + \frac{1}{15}C \le 8 \\ & & 0 \le B, C \end{array} \qquad \begin{array}{lll} \mathcal{D} & \text{minimize} & 1800R + 8L \\ & \text{subject to} & 20R + \frac{1}{15}L \ge 25 \\ & & 12R + \frac{1}{15}L \ge 20 \\ & & 0 \le R, L \end{array}$$

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$$(B, C)^* = (45, 75),$$

Plastic Cup Factory Reprised

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$$(B, C)^* = (45, 75), \quad (R, L)^* = \left(\frac{5}{8}, \frac{375}{2}\right),$$

Plastic Cup Factory Reprised

$$\begin{array}{llll} \mathcal{P} & \text{maximize} & 25B + 20C \\ & \text{subject to} & 20B + 12C \le 1800 \\ & & \frac{1}{15}B + \frac{1}{15}C \le 8 \\ & & 0 \le B, C \end{array} \qquad \begin{array}{llll} \mathcal{D} & \text{minimize} & 1800R + 8L \\ & \text{subject to} & 20R + \frac{1}{15}L \ge 25 \\ & & 12R + \frac{1}{15}L \ge 20 \\ & & 0 \le R, L \end{array}$$

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$$(B, C)^* = (45, 75), \quad (R, L)^* = \left(\frac{5}{8}, \frac{375}{2}\right), \quad z^* = 2625$$

Another Duality Example

$$
\begin{array}{lll}
\mathcal{P} & \max & 4x_1 + 5x_2 + 3x_3 \\
& \text{s.t.} & x_1 + 4x_2 + 2x_3 \quad \leq 11 \\
& & 3x_1 + 2x_2 + x_3 \quad \leq 5 \\
& & 4x_1 + 2x_2 + 2x_3 \quad \leq 8 \\
& & 0 \leq x_1, x_2, x_3
\end{array}
$$

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$$\begin{array}{llll} \mathcal{D} & \min & 11y_1 + 5y_2 + 8y_3 & \\ & \text{s.t.} & y_1 + 3y_2 + 4y_3 & \geq 4 \\ & & 4y_1 + 2y_2 + 2y_3 & \geq 5 \\ & & 2y_1 + y_2 + 2y_3 & \geq 3 \\ & & 0 \leq y_1, y_2, y_3 & \end{array}$$

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\qquad
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& & 2y_1 + y_2 + 2y_3 & \geq 3 \\
& & 0 \leq y_1, y_2, y_3 &
\end{array}
$$

$$
T_0 = \left[\begin{array}{cccccc|c}
1 & 4 & 2 & 1 & 0 & 0 & 11 \\
3 & 2 & 1 & 0 & 1 & 0 & 5 \\
4 & 2 & 2 & 0 & 0 & 1 & 8 \\
\hline
4 & 5 & 3 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Another Duality Example

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$$T_{\text{opt}} = \left[\begin{array}{cccccc|c} -5 & 0 & 0 & 1 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & \frac{-1}{2} & 1 \\ 1 & 0 & 1 & 0 & -1 & 1 & 3 \\ \hline -4 & 0 & 0 & 0 & -2 & \frac{-1}{2} & -14 \end{array}\right]$$

Another Duality Example

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$$x^* = (0, 1, 3), \quad y^* = (0, 2, 1/2), \quad z^* = 14$$

Another Duality Example

Check

$$
\begin{array}{llll}
\mathcal{P} & \max & 4x_1 + 5x_2 + 3x_3 & \\
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 & & 0 \le x_1, x_2, x_3 &
\end{array}
\qquad
\begin{array}{llll}
\mathcal{D} & \min & 11y_1 + 5y_2 + 8y_3 & \\
 & \text{s.t.} & y_1 + 3y_2 + 4y_3 & \ge 4 \\
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 & & 0 \le y_1, y_2, y_3 &
\end{array}
$$

$$x^* = (0, 1, 3), \quad y^* = (0, 2, 1/2), \quad z^* = 14$$

Strong Duality

If we can now show that the simplex algorithm works, then we have an algorithm that simultaneously solves both the primal and dual problems.

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If we can now show that the simplex algorithm works, then we have an algorithm that simultaneously solves both the primal and dual problems.

Moreover, the optimal value in the primal and dual coincides giving equality in the weak duality inequality.

We now focus on the details of the simplex algorithm to determine if and when it works.