Math 407: Linear Optimization

Lecture 5: Simplex Algorithm I

The Simplex Algorithm

We develop a method for solving standard form LPs by considering the following example.

$$\begin{array}{rrcl} \max & 5x_1 + 4x_2 + 3x_3 & & \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\ & 4x_1 + x_2 + 2x_3 & \leq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & 0 \leq x_1, x_2, x_3 & & \end{array}$$

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At this point we only have one tool for attacking linear systems. Gaussian elimination, a method for solving linear systems of equations. Let's try to use it to solve LPs.

Dictionaries

We must first build a linear system of equations that encodes all of the information associated with the LP.

$$\begin{array}{rrcl} \max & 5x_1 + 4x_2 + 3x_3 & & \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\ & 4x_1 + x_2 + 2x_3 & \leq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & 0 \leq x_1, x_2, x_3 & & \end{array}$$

Slack Variables and Dictionaries

For each linear inequality we introduce a new variable, called a *slack variable*, so that we can write each linear inequality as an equation.

$$\begin{array}{rcrcll} x_4 & = & 5 & - & [2x_1 + 3x_2 + x_3] & \geq 0, \\ x_5 & = & 11 & - & [4x_1 + x_2 + 2x_3] & \geq 0, \\ x_6 & = & 8 & - & [3x_1 + 4x_2 + 2x_3] & \geq 0. \end{array}$$

Slack Variables: x_4, x_5, x_6

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Slack Variables: x_4, x_5, x_6

Next we introduce a variable to represent the objective.

$$z = 5x_1 + 4x_2 + 3x_3.$$

Slack Variables and Dictionaries

For each linear inequality we introduce a new variable, called a *slack variable*, so that we can write each linear inequality as an equation.

$$\begin{array}{rcrcll} x_4 & = & 5 & - & [2x_1 + 3x_2 + x_3] & \geq 0, \\ x_5 & = & 11 & - & [4x_1 + x_2 + 2x_3] & \geq 0, \\ x_6 & = & 8 & - & [3x_1 + 4x_2 + 2x_3] & \geq 0. \end{array}$$

Slack Variables: x_4, x_5, x_6

Next we introduce a variable to represent the objective.

$$z = 5x_1 + 4x_2 + 3x_3.$$

This system of equations is called a *dictionary* for the the LP.
The slack variables and the objective are defined by the original decision variables.

Dictionaries, Augmented Matrices, the Simplex Tableau

The LP is now encoded as the system

$$\begin{array}{rcccccl}
2x_1 + 3x_2 + x_3 & + & x_4 & & & = & 5 \\
4x_1 + x_2 + 2x_3 & & & + \; x_5 & & = & 11 \\
3x_2 + 4x_2 + 2x_3 & & & & + \; x_6 & = & 8 \\
-z + 5x_1 + 4x_2 + 3x_3 & & & & & = & 0
\end{array}$$

$$0 \leq x_1, x_2, x_3, x_4, x_5, x_6.$$

Dictionaries, Augmented Matrices, the Simplex Tableau

The associated augmented matrix is

$$\left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ -1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

with $0 \leq x_1, x_2, x_3, x_4, x_5, x_6$.

Dictionaries, Augmented Matrices, the Simplex Tableau

The associated augmented matrix is

$$\left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline -1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

with $0 \leq x_1, x_2, x_3, x_4, x_5, x_6$.

This augmented matrix is called the initial *simplex tableau*.
The simplex tableau is nothing more than an augmented matrix for the linear system that relates all of the LP variables.

Basic and Nonbasic Variables

Recall the initial dictionary for our LP:

$$\begin{array}{rcrrrr} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ z & = & & 5x_1 & +4x_2 & +3x_3 \end{array}$$

We call the decision variables on the left (x_4, x_4, x_6) the *basic* variables, and those on the right the *nonbasic* variables (x_1, x_2, x_3).

Basic and Nonbasic Variables

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Basic Solutions Identified by Dictionaries

We think of the nonbasic variables as taking the value zero. This determines the value of the basic variables and the objective z.

$$\begin{array}{rcrrrr} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ z & = & & 5x_1 & +4x_2 & +3x_3 \end{array}$$

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Nonbasic: $x_1 = x_2 = x_3 = 0$
Basic: $x_4 = 5,\ x_5 = 11,\ x_6 = 8,\ z = 0$

Basic Solutions Identified by Dictionaries

We think of the nonbasic variables as taking the value zero. This determines the value of the basic variables and the objective z.

$$\begin{array}{rcrrrr} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ z & = & & 5x_1 & +4x_2 & +3x_3 \end{array}$$

Nonbasic: $x_1 = x_2 = x_3 = 0$
Basic: $x_4 = 5,\ x_5 = 11,\ x_6 = 8,\ z = 0$

This is called the *basic solution* associated with this dictionary.

Basic Feasible Solutions (BFS)

The basic solution

$$x_1 = x_2 = x_3 = 0 \quad x_4 = 5,\ x_5 = 11,\ x_6 = 8$$

is feasible for the LP

$$\begin{array}{rrcl} \max & 5x_1 + 4x_2 + 3x_3 & & \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\ & 4x_1 + x_2 + 2x_3 & \leq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & 0 \leq x_1, x_2, x_3 & & \end{array}$$

Such basic solutions are called *basic feasible solutions* (BFS).
The associated dictionary is said to be a *feasible dictionary*.

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Such basic solutions are called *basic feasible solutions* (BFS).
The associated dictionary is said to be a *feasible dictionary*.
In particular, this LP is said to have feasible origin, i.e. $(x_1, x_2, x_3) = (0, 0, 0)$ is feasible.

Grand Strategy: Pivoting

Move from one feasible dictionary to another increasing the value of the objective each time.

$$\begin{array}{rcrrrr} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ z & = & & 5x_1 & +4x_2 & +3x_3 \end{array}$$

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We do this by choosing a nonbasic variable with a positive coefficient, and then increase its value from zero as much as we can while maintaining feasibility.

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Let us increase the value of x_1 from zero.
How much can we increase x_1 and keep all other variables non-negative?

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How much can we increase x_1 and keep all other variables non-negative?
Consider the equation

$$0 \leq x_4 = 5 - 2x_1 - 3x_2 - x_3 \ .$$

Pivoting

Let us increase the value of x_1 from zero.
How much can we increase x_1 and keep all other variables non-negative?
Consider the equation

$$0 \leq x_4 = 5 - 2x_1 - 3x_2 - x_3 \ .$$

Keeping x_4 non-negative implies that we cannot increase x_1 by more than $5/2$. Any further increase will push x_4 negative.

Pivoting

Next consider the variable x_5:

$$0 \leq x_5 = 11 - 4x_1 - x_2 - 2x_3 .$$

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Similarly, for x_6 we have

$$0 \leq x_6 = 8 - 3x_1 - 4x_2 - 2x_3 \ .$$

Therefore, we cannot increase x_1 by more than $8/3$.

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$$0 \leq x_5 = 11 - 4x_1 - x_2 - 2x_3 .$$

Therefore, we cannot increase x_1 by more than $11/4$.

Similarly, for x_6 we have

$$0 \leq x_6 = 8 - 3x_1 - 4x_2 - 2x_3 .$$

Therefore, we cannot increase x_1 by more than $8/3$.
Hence, we must have

$$x_1 \leq \min\{5/2, 11/4, 8/3\} = 5/2 .$$

Pivoting

$$\begin{array}{rcrrrr} x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ z & = & & 5x_1 & +4x_2 & +3x_3 \end{array}$$

Pivoting

$$\begin{array}{rcrrrrcl}
 & & & & & & \text{ratios} & \\
x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 & 5/2 & \leftarrow \text{ smallest ratio} \\
x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 & 11/4 & \\
x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 & 8/3 & \\
z & = & & 5x_1 & +4x_2 & +3x_3 & &
\end{array}$$

Pivoting

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If we increase x_1 to 5/2, then x_4 decreases to zero.
In this case we say x_4 leaves the basis and x_1 enters.

Pivoting

$$\begin{array}{rcrrrrcl}
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x_4 & = & 5 & -2x_1 & -3x_2 & -x_3 & 5/2 & \leftarrow \text{ smallest ratio} \\
x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 & 11/4 & \\
x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 & 8/3 & \\
z & = & & 5x_1 & +4x_2 & +3x_3 & &
\end{array}$$

If we increase x_1 to $5/2$, then x_4 decreases to zero.
In this case we say x_4 leaves the basis and x_1 enters.
Moving x_4 to the rhs and x_1 to the lhs gives

$$x_1 = 5/2 - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3.$$

Pivoting

We now have

$$\begin{array}{rcrrrr} x_1 & = & (5/2) & -(1/2)x_4 & -(3/2)x_2 & -(1/2)x_3 \\ x_5 & = & 11 & -4x_1 & -x_2 & -2x_3 \\ x_6 & = & 8 & -3x_1 & -4x_2 & -2x_3 \\ z & = & & 5x_1 & +4x_2 & +3x_3 \end{array}$$

Use the first equation to remove x_1 from the rhs to recover a dictionary.

Pivoting

$$\begin{aligned}
x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\
x_5 &= 11 - 4\left(\frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) - x_2 - 2x_3 \\
&= 1 + 2x_4 + 5x_2 \\
x_6 &= 8 - 3\left(\frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) - 4x_2 - 2x_3 \\
&= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\
z &= 5\left(\frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) + 4x_2 + 3x_3 \\
&= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3.
\end{aligned}$$

New Dictionary

$$\begin{aligned}
x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\
x_5 &= 1 + 2x_4 + 5x_2 \\
x_6 &= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\
z &= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3,
\end{aligned}$$

New Dictionary

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\ x_5 &= 1 + 2x_4 + 5x_2 \\ x_6 &= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ z &= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3, \end{aligned}$$

New BFS

Nonbasic Variables: $x_4 = x_2 = x_3 = 0$

Basic Variables: $x_1 = 5/2,\ x_5 = 1,\ x_6 = 1/2$

Objective value: $z = 25/2$

The Second Pivot

$$\begin{array}{rcrrrr}
x_1 & = & \frac{5}{2} & -\frac{1}{2}x_4 & -\frac{3}{2}x_2 & -\frac{1}{2}x_3 \\
x_5 & = & 1 & +2x_4 & +5x_2 & \\
x_6 & = & \frac{1}{2} & +\frac{3}{2}x_4 & +\frac{1}{2}x_2 & -\frac{1}{2}x_3 \\
\\
z & = & \frac{25}{2} & -\frac{5}{2}x_4 & -\frac{7}{2}x_2 & +\frac{1}{2}x_3
\end{array}$$

The Second Pivot

$$\begin{array}{rcrrrrc}
 & & & & & & \text{ratios}\\
x_1 & = & \frac{5}{2} & -\frac{1}{2}x_4 & -\frac{3}{2}x_2 & -\frac{1}{2}x_3 & 5\\
x_5 & = & 1 & +2x_4 & +5x_2 & & \\
x_6 & = & \frac{1}{2} & +\frac{3}{2}x_4 & +\frac{1}{2}x_2 & -\frac{1}{2}x_3 & 1\\
 & & & & & & \\
z & = & \frac{25}{2} & -\frac{5}{2}x_4 & -\frac{7}{2}x_2 & +\frac{1}{2}x_3 & \\
 & & & & & \uparrow & \\
 & & & & & pos. &
\end{array}$$

The Second Pivot

$$\begin{array}{rcllll l l}
 & & & & & & \text{ratios} & \\
x_1 & = & \frac{5}{2} & -\frac{1}{2}x_4 & -\frac{3}{2}x_2 & -\frac{1}{2}x_3 & 5 & \\
x_5 & = & 1 & +2x_4 & +5x_2 & & & \\
x_6 & = & \frac{1}{2} & +\frac{3}{2}x_4 & +\frac{1}{2}x_2 & -\frac{1}{2}x_3 & 1 & \rightarrow \text{smallest} \\
 & & & & & & & \\
z & = & \frac{25}{2} & -\frac{5}{2}x_4 & -\frac{7}{2}x_2 & +\frac{1}{2}x_3 & & \\
 & & & & & \uparrow & & \\
 & & & & & pos. & &
\end{array}$$

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$$\begin{array}{rcrrrrccl}
 & & & & & & \text{ratios} & & \\
x_1 & = & \frac{5}{2} & -\frac{1}{2}x_4 & -\frac{3}{2}x_2 & -\frac{1}{2}x_3 & 5 & & \\
x_5 & = & 1 & +2x_4 & +5x_2 & & & & \\
x_6 & = & \frac{1}{2} & +\frac{3}{2}x_4 & +\frac{1}{2}x_2 & -\frac{1}{2}x_3 & 1 & \to & \text{smallest} \\
 & & & & & & & & \\
z & = & \frac{25}{2} & -\frac{5}{2}x_4 & -\frac{7}{2}x_2 & +\frac{1}{2}x_3 & & & \\
 & & & & & \uparrow & & & \\
 & & & & & \textit{pos.} & & &
\end{array}$$

x_3 enters the basis and x_6 leaves.

$$x_3 = 1 + 3x_4 + x_2 - 2x_6$$

The Second Pivot

The new dictionary is

$$\begin{aligned} x_3 &= 1 + 3x_4 + x_2 - 2x_6 \\ x_1 &= 2 - 2x_4 - 2x_2 + x_6 \\ x_5 &= 1 + 2x_4 + 2x_2 \\ z &= 13 - x_4 - 3x_2 - x_6 \end{aligned}$$

The Second Pivot

The new dictionary is

$$\begin{aligned} x_3 &= 1 + 3x_4 + x_2 - 2x_6 \\ x_1 &= 2 - 2x_4 - 2x_2 + x_6 \\ x_5 &= 1 + 2x_4 + 2x_2 \\ z &= 13 - x_4 - 3x_2 - x_6 \end{aligned}$$

The BFS identified by this dictionary is
Nonbasic variables: $x_4 = x_2 = x_6 = 0$
Basic variables: $x_1 = 2,\ x_3 = 1,\ x_5 = 1$
Objective value: $z = 13$

The Third Pivot

$$\begin{aligned} x_3 &= 1 + 3x_4 + x_2 - 2x_6 \\ x_1 &= 2 - 2x_4 - 2x_2 + x_6 \\ x_5 &= 1 + 2x_4 + 2x_2 \\ z &= 13 - x_4 - 3x_2 - x_6 \end{aligned}$$

What nonbasic variable should enter the basis?

The Third Pivot

$$\begin{aligned} x_3 &= 1 + 3x_4 + x_2 - 2x_6 \\ x_1 &= 2 - 2x_4 - 2x_2 + x_6 \\ x_5 &= 1 + 2x_4 + 2x_2 \\ z &= 13 - x_4 - 3x_2 - x_6 \end{aligned}$$

What nonbasic variable should enter the basis?
No candidate! All have a negative coefficient in the z row.
If we increase the value of any nonbasic variable, the value of the objective will be reduced.
What does this mean?

The Third Pivot

$$\begin{aligned} x_3 &= 1 + 3x_4 + x_2 - 2x_6 \\ x_1 &= 2 - 2x_4 - 2x_2 + x_6 \\ x_5 &= 1 + 2x_4 + 2x_2 \\ z &= 13 - x_4 - 3x_2 - x_6 \end{aligned}$$

What nonbasic variable should enter the basis?
No candidate! All have a negative coefficient in the z row.
If we increase the value of any nonbasic variable, the value of the objective will be reduced.
What does this mean? **The current BFS is optimal!**

Optimal BFS and Dictionary

The optimal dictionary is

$$\begin{aligned} x_3 &= 1 + 3x_4 + x_2 - 2x_6 \\ x_1 &= 2 - 2x_4 - 2x_2 + x_6 \\ x_5 &= 1 + 2x_4 + 2x_2 \\ z &= 13 - x_4 - 3x_2 - x_6 \end{aligned}$$

Optimal BFS and Dictionary

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$$\begin{array}{rcl} x_3 & = & 1 + 3x_4 + x_2 - 2x_6 \\ x_1 & = & 2 - 2x_4 - 2x_2 + x_6 \qquad\qquad \text{feasible} \\ x_5 & = & 1 + 2x_4 + 2x_2 \\ z & = & 13 - x_4 - 3x_2 - x_6 \qquad \rightarrow \ \text{all neg. coef.s} \end{array}$$

Optimal BFS and Dictionary

The optimal dictionary is

$$\begin{array}{rcll} x_3 & = & 1 + 3x_4 + x_2 - 2x_6 & \\ x_1 & = & 2 - 2x_4 - 2x_2 + x_6 & \qquad \text{feasible} \\ x_5 & = & 1 + 2x_4 + 2x_2 & \\ z & = & 13 - x_4 - 3x_2 - x_6 & \qquad \rightarrow \text{ all neg. coef.s} \end{array}$$

The optimal BFS is $x = (2, 0, 1, 0, 1, 0)^T$.
The optimal value is $z = 13$.

The Simplex Algorithm

The process of moving from one feasible dictionary to another is called *simplex pivoting*.

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A pivot corresponds to doing Gauss-Jordan elimination on the column in the simplex tableau (augmented matrix) corresponding to the incoming variable.
Hence the simplex algorithm is a process that can be applied directly to the simplex tableau.

Simplex Pivoting on the Augmented Matrix

$$\begin{array}{rrcl} \max & 5x_1 + 4x_2 + 3x_3 & & \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 & \leq & 5 \\ & 4x_1 + x_2 + 2x_3 & \leq & 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\ & 0 \leq x_1, x_2, x_3 & & \end{array}$$

The linear system associated with the initial dictionary is given by

$$\begin{array}{rcccccccl} 2x_1 + 3x_2 + x_3 & + & x_4 & & & & & = & 5 \\ 4x_1 + x_2 + 2x_3 & & & + & x_5 & & & = & 11 \\ 3x_2 + 4x_2 + 2x_3 & & & & & + & x_6 & = & 8 \\ -z + 5x_1 + 4x_2 + 3x_3 & & & & & & & = & 0 \end{array}$$

$$0 \leq x_1, x_2, x_3, x_4, x_5, x_6.$$

Simplex Pivoting on the Augmented Matrix

$$\begin{array}{rcccccl} 2x_1+3x_2+x_3 & + & x_4 & & & = & 5\\ 4x_1+x_2+2x_3 & & + & x_5 & & = & 11\\ 3x_2+4x_2+2x_3 & & & + & x_6 & = & 8\\ -z+5x_1+4x_2+3x_3 & & & & & = & 0 \end{array}$$

$$0 \leq x_1, x_2, x_3, x_4, x_5, x_6.$$

Simplex Pivoting on the Augmented Matrix

$$\begin{array}{rcccccl}
2x_1 + 3x_2 + x_3 & + & x_4 & & & = & 5 \\
4x_1 + x_2 + 2x_3 & & & + \quad x_5 & & = & 11 \\
3x_2 + 4x_2 + 2x_3 & & & & + \quad x_6 & = & 8 \\
-z + 5x_1 + 4x_2 + 3x_3 & & & & & = & 0
\end{array}$$

$$0 \le x_1, x_2, x_3, x_4, x_5, x_6.$$

The augmented matrix associated with the initial dictionary is

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right] = \left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline -1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

Simplex Pivoting on the Augmented Matrix

$$\begin{array}{rcccccl} 2x_1 + 3x_2 + x_3 & + & x_4 & & & = & 5 \\ 4x_1 + x_2 + 2x_3 & & & + & x_5 & & = 11 \\ 3x_2 + 4x_2 + 2x_3 & & & & + \quad x_6 & = & 8 \\ -z + 5x_1 + 4x_2 + 3x_3 & & & & & = & 0 \end{array}$$

$$0 \leq x_1, x_2, x_3, x_4, x_5, x_6.$$

The augmented matrix associated with the initial dictionary is

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right] = \begin{array}{c} \begin{array}{ccccccc} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline -1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Simplex Pivoting on the Augmented Matrix

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right] = \left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline -1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

Simplex Pivoting on the Augmented Matrix

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right] = \left[\begin{array}{rrrrrrr|r} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline -1 & \underset{\displaystyle\uparrow}{5} & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

Simplex Pivoting on the Augmented Matrix

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right] = \begin{array}{ll} \left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline -1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{cc} \text{ratios} & \\ 5/2 & \leftarrow \\ 11/4 & \\ 8/3 & \\ & \end{array} \\ \begin{array}{cccccccc} & \uparrow & & & & & & \end{array} & \end{array}$$

Simplex Pivoting on the Augmented Matrix

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right] = \left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline -1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{cc} \text{ratios} & \\ 5/2 & \leftarrow \\ 11/4 & \\ 8/3 & \\ & \end{array}$$

$$\begin{array}{cccccccc} & \uparrow & & & \uparrow & \uparrow & \uparrow & \end{array}$$

Which variables are in the basis?

Simplex Pivoting on the Augmented Matrix

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right] = \left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline -1 & \underset{\uparrow}{5} & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{cc} \text{ratios} & \\ 5/2 & \leftarrow \\ 11/4 & \\ 8/3 & \\ & \end{array}$$

Which variables are in the basis?
Columns of the identity.

Simplex Pivoting on the Augmented Matrix

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right] = \left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline -1 & \underset{\uparrow}{5} & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{cl} \text{ratios} & \\ 5/2 & \leftarrow \\ 11/4 & \\ 8/3 & \\ & \end{array}$$

How do we choose the variable to enter the basis?

Simplex Pivoting on the Augmented Matrix

$$\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right] = \begin{array}{c} \left[\begin{array}{ccccccc|c} 0 & 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline -1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccccc} & \uparrow & & & & & & \end{array} \end{array} \begin{array}{cc} \text{ratios} & \\ 5/2 & \leftarrow \\ 11/4 & \\ 8/3 & \\ & \\ & \end{array}$$

How do we choose the variable to enter the basis?
How do we choose the variable to leave the basis?

Simplex Tableau Pivot

$$\begin{array}{ccccccc|c c}
 & & & & & & & & \text{ratios} \\
 & \downarrow & & & & & & & \\
0 & \textcircled{2} & 3 & 1 & 1 & 0 & 0 & 5 & \textcircled{5/2} \\
0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 & 11/4 \\
0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 & 8/3 \\
\hline
-1 & \textcircled{5} & 4 & 3 & 0 & 0 & 0 & 0 & \\
\hline
\end{array}$$

Simplex Tableau Pivot

$$\begin{array}{ccccccc|cc}
 & \text{Pivot column} & & & & & & & \text{ratios}\\
 & \downarrow & & & & & & & \\
0 & \textcircled{2} & 3 & 1 & 1 & 0 & 0 & 5 & \textcircled{5/2}\\
0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 & 11/4\\
0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 & 8/3\\
\cline{1-8}
-1 & \textcircled{5} & 4 & 3 & 0 & 0 & 0 & 0 & \\
\cline{1-8}
\end{array}$$

Simplex Tableau Pivot

$$\begin{array}{ccccccc|cc}
 & \begin{array}{c}\text{Pivot}\\ \text{column}\\ \downarrow\end{array} & & & & & & & \text{ratios}\\
0 & \textcircled{2} & 3 & 1 & 1 & 0 & 0 & 5 & \textcircled{5/2}\\
0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 & 11/4\\
0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 & 8/3\\
\hline
-1 & \textcircled{5} & 4 & 3 & 0 & 0 & 0 & 0 & \\
\hline
\end{array}$$

Simplex Tableau Pivot

$$\begin{array}{ccccccc|c|cl}
 & \begin{array}{c}\text{Pivot}\\ \text{column}\\ \downarrow\end{array} & & & & & & & \text{ratios} & \\
0 & \boxed{2} & 3 & 1 & 1 & 0 & 0 & 5 & \boxed{5/2} & \leftarrow \quad \text{Pivot row} \\
0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 & 11/4 & \\
0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 & 8/3 & \\
\hline
-1 & \boxed{5} & 4 & 3 & 0 & 0 & 0 & 0 & & \\
\hline
\end{array}$$

Simplex Tableau Pivot

$$\begin{array}{ccccccc|cccc}
 & \text{Pivot column} & & & & & & & \text{ratios} & & \\
 & \downarrow & & & & & & & & & \\
0 & \enclose{circle}{2} & 3 & 1 & 1 & 0 & 0 & 5 & \enclose{circle}{5/2} & \leftarrow & \text{Pivot row} \\
0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 & 11/4 & & \\
0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 & 8/3 & & \\ \hline
-1 & \enclose{circle}{5} & 4 & 3 & 0 & 0 & 0 & 0 & & & \\ \hline
 & & & & & & & & & & \\ \hline
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 & & & \\
 & & & & & & & & & & \\ \hline
 & & & & & & & & & & \\ \hline
\end{array}$$

Simplex Tableau Pivot

$$
\begin{array}{rrrrrrr|r cc}
 & \overset{\text{Pivot column}}{\downarrow} & & & & & & & \text{ratios} & \\
0 & \textcircled{2} & 3 & 1 & 1 & 0 & 0 & 5 & \textcircled{5/2} & \leftarrow \text{Pivot row} \\
0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 & 11/4 & \\
0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 & 8/3 & \\
\hline
-1 & \textcircled{5} & 4 & 3 & 0 & 0 & 0 & 0 & & \\
\hline
\end{array}
$$

$$
\begin{array}{rrrrrrr|r}
\hline
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
 & & & & & & & \\
\hline
 & & & & & & & \\
\hline
\end{array}
$$

Simplex Tableau Pivot

$$
\begin{array}{ccccccc|c|ccc}
 & \text{Pivot column} & & & & & & & \text{ratios} & & \\
 & \downarrow & & & & & & & & & \\
0 & \textcircled{2} & 3 & 1 & 1 & 0 & 0 & 5 & \textcircled{5/2} & \leftarrow & \text{Pivot row} \\
0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 & 11/4 & & \\
0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 & 8/3 & & \\
\hline
-1 & \textcircled{5} & 4 & 3 & 0 & 0 & 0 & 0 & & & \\
\hline
\end{array}
$$

$$
\begin{array}{ccccccc|c}
\hline
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 \\
\hline
 & & & & & & & \\
\hline
\end{array}
$$

Simplex Tableau Pivot

$$\begin{array}{ccccccc|cccc}
 & \begin{array}{c}\text{Pivot}\\ \text{column}\\ \downarrow\end{array} & & & & & & & \text{ratios} & & \\
0 & \enclose{circle}{2} & 3 & 1 & 1 & 0 & 0 & 5 & \enclose{circle}{5/2} & \leftarrow & \text{Pivot row}\\
0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 & 11/4 & & \\
0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 & 8/3 & & \\
\hline
\text{-1} & \enclose{circle}{5} & 4 & 3 & 0 & 0 & 0 & 0 & & & \\
\hline
\end{array}$$

$$\begin{array}{ccccccc|c}
\hline
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2\\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1\\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2\\
\hline
 & & & & & & & \\
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2\\
\hline
\end{array}$$

Simplex Tableau and Its Dictionary

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\ 0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 \\ \hline -1 & 0 & -7/2 & 1/2 & -5/2 & 0 & & -25/2 \end{array}\right],$$

This tableau is the augmented matrix for the dictionary

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\ x_5 &= 1 + 2x_4 + 5x_2 \\ x_6 &= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ z &= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3, \end{aligned}$$

Simplex Tableau and Its Dictionary

$$\left[\begin{array}{ccccccc|c}
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & & -25/2
\end{array}\right],$$

Simplex Tableau and Its Dictionary

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\ 0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 \\ \hline -1 & 0 & -7/2 & 1/2 & -5/2 & 0 & & -25/2 \end{array}\right],$$

$$x_1 \quad = \quad \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3$$

Simplex Tableau and Its Dictionary

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\ 0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 \\ \hline -1 & 0 & -7/2 & 1/2 & -5/2 & 0 & & -25/2 \end{array}\right] ,$$

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\ x_5 &= 1 + 2x_4 + 5x_2 \end{aligned}$$

Simplex Tableau and Its Dictionary

$$\left[\begin{array}{ccccccc|c}
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & & -25/2
\end{array}\right],$$

$$\begin{aligned}
x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\
x_5 &= 1 + 2x_4 + 5x_2 \\
x_6 &= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3
\end{aligned}$$

Simplex Tableau and Its Dictionary

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\ 0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 \\ \hline -1 & 0 & -7/2 & 1/2 & -5/2 & 0 & & -25/2 \end{array}\right],$$

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\ x_5 &= 1 + 2x_4 + 5x_2 \\ x_6 &= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ z &= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3, \end{aligned}$$

Simplex Tableau and Its Dictionary

$$\left[\begin{array}{ccccccc|c}
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & & -25/2
\end{array}\right],$$

$$\begin{array}{rcl}
x_1 & = & \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\
x_5 & = & 1 + 2x_4 + 5x_2 \\
x_6 & = & \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\
z & = & \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3,
\end{array}$$

The BFS is obtained by setting non basic variable equal to zero.

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (5/2,\ 0, 0, 0,\ 1,\ 1/2).$$

Second Simplex Pivot on the Tableau

$$
\begin{array}{ccccccc|c}
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 \\
\hline
\end{array}
$$

Second Simplex Pivot on the Tableau

$$\begin{array}{ccccccc|c}
 & & & \begin{array}{c}\text{Pivot}\\ \text{column}\\ \downarrow\end{array} & & & & \\
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2\\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1\\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2\\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2\\
\hline
\end{array}$$

Second Simplex Pivot on the Tableau

$$\begin{array}{ccccccc|cc}
 & & & \begin{array}{c}\text{Pivot}\\ \text{column}\\ \downarrow\end{array} & & & & & \text{ratios}\\
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 & 5\\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 & \\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 & 1\\
\cline{1-8}
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 & \\
\cline{1-8}
\end{array}$$

Second Simplex Pivot on the Tableau

$$\begin{array}{ccccccc|cccc}
 & & & \text{Pivot column} & & & & & \text{ratios} & & \\
 & & & \downarrow & & & & & & & \\
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 & 5 & & \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 & & & \\
0 & 0 & -1/2 & 1/2 & -3/2 & 0 & 1 & 1/2 & 1 & \leftarrow & \text{pivot row} \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 & & & \\
\hline
\end{array}$$

Second Simplex Pivot on the Tableau

$$
\begin{array}{ccccccc|cc}
 & & & & & & & & \text{ratios} \\
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 & 5 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 & \\
0 & 0 & -1/2 & \textcircled{1/2} & -3/2 & 0 & 1 & 1/2 & 1 \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 & \\
\hline
\end{array}
$$

Second Simplex Pivot on the Tableau

								ratios
0	1	$3/2$	$1/2$	$1/2$	0	0	$5/2$	5
0	0	-5	0	-2	1	0	1	
0	0	$-1/2$	$\boxed{1/2}$	$-3/2$	0	1	$1/2$	1
-1	0	$-7/2$	$1/2$	$-5/2$	0	0	$-25/2$	

0	0	-1	1	-3	0	2	1

Second Simplex Pivot on the Tableau

$$\begin{array}{ccccccc|cc}
 & & & & & & & & \text{ratios} \\
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 & 5 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 & \\
0 & 0 & -1/2 & \enclose{circle}{1/2} & -3/2 & 0 & 1 & 1/2 & 1 \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 & \\
\hline
 & & & & & & & & \\
\hline
0 & 1 & 2 & 0 & 2 & 0 & -1 & 2 & \\
 & & & & & & & & \\
0 & 0 & -1 & 1 & -3 & 0 & 2 & 1 & \\
\hline
 & & & & & & & & \\
\hline
\end{array}$$

Second Simplex Pivot on the Tableau

$$\begin{array}{ccccccc|cc}
 & & & & & & & & \text{ratios} \\
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 & 5 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 & \\
0 & 0 & -1/2 & \textcircled{1/2} & -3/2 & 0 & 1 & 1/2 & 1 \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 & \\
\hline
\end{array}$$

$$\begin{array}{ccccccc|c}
\hline
0 & 1 & 2 & 0 & 2 & 0 & -1 & 2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -1 & 1 & -3 & 0 & 2 & 1 \\
\hline
 & & & & & & & \\
\hline
\end{array}$$

Second Simplex Pivot on the Tableau

$$\begin{array}{ccccccc|cc}
 & & & & & & & & \text{ratios} \\
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 & 5 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 & \\
0 & 0 & -1/2 & \textcircled{$1/2$} & -3/2 & 0 & 1 & 1/2 & 1 \\
\cline{1-8}
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 & \\
\cline{1-8}
\end{array}$$

$$\begin{array}{ccccccc|c}
\hline
0 & 1 & 2 & 0 & 2 & 0 & -1 & 2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -1 & 1 & -3 & 0 & 2 & 1 \\
\hline
-1 & 0 & -3 & 0 & -1 & 0 & -1 & -13 \\
\hline
\end{array}$$

Second Simplex Pivot on the Tableau

								ratios
0	1	3/2	1/2	1/2	0	0	5/2	5
0	0	−5	0	−2	1	0	1	
0	0	−1/2	$\boxed{1/2}$	−3/2	0	1	1/2	1
−1	0	−7/2	1/2	−5/2	0	0	−25/2	

0	1	2	0	2	0	−1	2
0	0	−5	0	−2	1	0	1
0	0	−1	1	−3	0	2	1
−1	0	−3	0	−1	0	−1	−13

Optimal Solution: $(x_1, x_2, x_3) = (2, 0, 1)$

Second Simplex Pivot on the Tableau

$$\begin{array}{ccccccc|cc}
 & & & & & & & & \text{ratios} \\
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 & 5 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 & \\
0 & 0 & -1/2 & \enclose{circle}{1/2} & -3/2 & 0 & 1 & 1/2 & 1 \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 & \\
\hline
\end{array}$$

$$\begin{array}{ccccccc|c}
\hline
0 & 1 & 2 & 0 & 2 & 0 & -1 & 2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -1 & 1 & -3 & 0 & 2 & 1 \\
\hline
-1 & 0 & -3 & 0 & -1 & 0 & -1 & -13 \\
\hline
\end{array}$$

Optimal Solution: $(x_1, x_2, x_3) = (2, 0, 1)$

Optimal Value: $z = 13$

Recap: Tableau Pivoting

$$\begin{array}{ccccccc|c}
0 & \textcircled{2} & 3 & 1 & 1 & 0 & 0 & 5 \\
0 & 4 & 1 & 2 & 0 & 1 & 0 & 11 \\
0 & 3 & 4 & 2 & 0 & 0 & 1 & 8 \\
\hline
-1 & 5 & 4 & 3 & 0 & 0 & 0 & 0 \\
\hline
\end{array}$$

$$\begin{array}{ccccccc|c}
\hline
0 & 1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -1/2 & \textcircled{1/2} & -3/2 & 0 & 1 & 1/2 \\
\hline
-1 & 0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 \\
\hline
\end{array}$$

$$\begin{array}{ccccccc|c}
\hline
0 & 1 & 2 & 0 & 2 & 0 & -1 & 2 \\
0 & 0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -1 & 1 & -3 & 0 & 2 & 1 \\
\hline
-1 & 0 & -3 & 0 & -1 & 0 & -1 & -13 \\
\hline
\end{array}$$

Remove z Column

$$\begin{array}{cccccc|c}
\enclose{circle}{2} & 3 & 1 & 1 & 0 & 0 & 5 \\
4 & 1 & 2 & 0 & 1 & 0 & 11 \\
3 & 4 & 2 & 0 & 0 & 1 & 8 \\
\hline
5 & 4 & 3 & 0 & 0 & 0 & 0 \\
\hline
\end{array}$$

$$\begin{array}{cccccc|c}
\hline
1 & 3/2 & 1/2 & 1/2 & 0 & 0 & 5/2 \\
0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & -1/2 & \enclose{circle}{1/2} & -3/2 & 0 & 1 & 1/2 \\
\hline
0 & -7/2 & 1/2 & -5/2 & 0 & 0 & -25/2 \\
\hline
\end{array}$$

$$\begin{array}{cccccc|c}
\hline
1 & 2 & 0 & 2 & 0 & -1 & 2 \\
0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & -1 & 1 & -3 & 0 & 2 & 1 \\
\hline
0 & -3 & 0 & -1 & 0 & -1 & -13
\end{array}$$

Another Example

$$\begin{array}{lrcrcrcl} \text{maximize} & 3x & + & 2y & - & 4z & & \\ \\ \text{subject to} & x & + & 4y & & & \leq & 5 \\ & 2x & + & 4y & - & 2z & \leq & 6 \\ & x & + & y & - & 2z & \leq & 2 \\ & 0 & \leq & x, & y, & z & & \end{array}$$

Second Example: Tableau Pivoting

$$\begin{array}{cccccc|cc}
 & & & & & & & \text{ratios} \\
1 & 4 & 0 & 1 & 0 & 0 & 5 & 5 \\
2 & 4 & -2 & 0 & 1 & 0 & 6 & 3 \\
\textcircled{1} & 1 & -2 & 0 & 0 & 1 & 2 & 2 \\
\cline{1-7}
3 & 2 & -4 & 0 & 0 & 0 & 0 & \\
\cline{1-7}
\end{array}$$

$$\begin{array}{cccccc|cc}
\cline{1-7}
0 & 3 & 2 & 1 & 0 & -1 & 3 & 3/2 \\
0 & 2 & \textcircled{2} & 0 & 1 & -2 & 2 & 1 \\
1 & 1 & -2 & 0 & 0 & 1 & 2 & \\
\cline{1-7}
0 & -1 & 2 & 0 & 0 & -3 & -6 & \\
\cline{1-7}
\end{array}$$

$$\begin{array}{cccccc|c}
\hline
0 & 1 & 0 & 1 & -1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1/2 & -1 & 1 \\
1 & 3 & 0 & 0 & 1 & -1 & 4 \\
\hline
0 & -3 & 0 & 0 & -1 & -1 & -8
\end{array}$$