

Math 407A: Linear Optimization

Lecture 4: LP Standard Form ¹

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LPs in Standard Form

Minimization \rightarrow maximization

Linear equations to linear inequalities

Lower and upper bounded variables

Interval variable bounds

Free variable

Two Step Process to Standard Form

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$0 \leq x$ All variables must be non-negative.

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If an LP has an inequality constraint of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i,$$

it can be transformed to one in standard form by multiplying the inequality through by -1 to get

$$-a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \leq -b_i.$$

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- ▶ *linear equations*

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► linear equations

The linear equation

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i$$

can be written as two linear inequalities

$$a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$$

and

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i.$$

or equivalently

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &\leq b_i \\ -a_{i1}x_1 - \cdots - a_{in}x_n &\leq -b_i \end{aligned}$$

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If a variable x_i has lower bound l_i which is not zero ($l_i \leq x_i$), one obtains a non-negative variable w_i with the substitution

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In this case, the bound $l_i \leq x_i$ is equivalent to the bound $0 \leq w_i$.

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If a variable x_i has an upper bound u_i ($x_i \leq u_i$) one obtains a non-negative variable w_i with the substitution

$$x_i = u_i - w_i.$$

In this case, the bound $x_i \leq u_i$ is equivalent to the bound $0 \leq w_i$.

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- ▶ *variables with interval bounds*

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An interval bound of the form $l_i \leq x_i \leq u_i$ can be transformed into one non-negativity constraint and one linear inequality constraint in standard form by making the substitution

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$$0 \leq w_i \quad \text{and} \quad w_i \leq u_i - l_i.$$

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Sometimes a variable is given without any bounds. Such variables are called free variables. To obtain standard form every free variable must be replaced by the difference of two non-negative variables. That is, if x_i is free, then we get

$$x_i = u_i - v_i$$

with $0 \leq u_i$ and $0 \leq v_i$.

Transformation to Standard Form

Put the following LP into standard form.

$$\begin{array}{llllllll} \text{minimize} & 3x_1 & - & x_2 & & & & \\ \text{subject to} & -x_1 & + & 6x_2 & - & x_3 & + & x_4 \geq -3 \\ & & & 7x_2 & & & + & x_4 = 5 \\ & & & & & x_3 & + & x_4 \leq 2 \end{array}$$

$$-1 \leq x_2, \quad x_3 \leq 5, \quad -2 \leq x_4 \leq 2.$$

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Step 1: Make all of the changes that do not involve a variable substitution.

Step 2: Make all of the variable substitutions.

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$$x_1 - 6x_2 + x_3 - x_4 \leq 3.$$

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$$x_4 \leq 2$$

Step 1: Transformation to Standard Form

Combining all of these changes gives the LP

$$\begin{array}{ll} \text{maximize} & -3x_1 + x_2 \\ \text{subject to} & x_1 - 6x_2 + x_3 - x_4 \leq 3 \\ & 7x_2 + x_4 \leq 5 \\ & -7x_2 - x_4 \leq -5 \\ & x_3 + x_4 \leq 2 \\ & x_4 \leq 2 \\ & -1 \leq x_2, \quad x_3 \leq 5, \quad -2 \leq x_4. \end{array}$$

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x_4 is bounded below ($-2 \leq x_4$), so we replace it by

$$z_4 = x_4 + 2 \quad \text{or} \quad x_4 = z_4 - 2 \quad \text{with} \quad 0 \leq z_4.$$

Step 2: Transformation to Standard Form

$$\begin{array}{ll} \text{maximize} & -3z_1^+ + 3z_1^- + x_2 \\ \text{subject to} & z_1^+ - z_1^- - 6x_2 + x_3 - x_4 \leq 3 \\ & 7x_2 + x_4 \leq 5 \\ & -7x_2 - x_4 \leq -5 \\ & x_3 + x_4 \leq 2 \\ & x_4 \leq 2 \end{array}$$

$$0 \leq z_1^+, 0 \leq z_1^-, -1 \leq x_2, x_3 \leq 5, -2 \leq x_4.$$

Step 2: Transformation to Standard Form

$$\begin{array}{l} \text{maximize} \\ \text{subject to} \end{array} \begin{array}{r} -3z_1^+ + 3z_1^- + z_2 \\ z_1^+ - z_1^- - 6z_2 - z_3 - x_4 \leq -3 - 5 = -8 \\ 7z_2 + x_4 \leq 12 \\ -7z_2 - x_4 \leq -12 \\ -z_3 + x_4 \leq 2 - 5 = -3 \\ x_4 \leq 2 \end{array}$$

$$0 \leq z_1^+, 0 \leq z_1^-, -0 \leq z_2, 0 \leq z_3, -2 \leq x_4.$$

Step 2: Transformation to Standard Form

After making these substitutions, we get the following LP in standard form:

Transformation to Standard Form: Practice

Transform the following LP to an LP in standard form.

$$\text{minimize} \quad x_1 - 12x_2 + 2x_3$$

$$\text{subject to} \quad -5x_1 - x_2 + 3x_3 = -15$$

$$2x_1 + x_2 - 20x_3 \geq -30$$

$$0 \leq x_2, \quad 1 \leq x_3 \leq 4$$