Math 407A: Linear Optimization

Lecture 2

Math Dept, University of Washington

January 7, 2010

Solving Systems of Linear equations

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Find all solutions $x \in \mathbb{R}^{n}$ to the system $Ax = b$.

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If a solution $x_0 \in \mathbb{R}^n$ exists, then the set of solutions is given by

$$x_0 + \mathrm{Nul}(A) .$$

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The three elementary row operations.

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2. Multiply any row by a non-zero constant.
3. Replace any row by itself plus a multiple of any *other* row.

These elementary row operations can be interpreted as multiplying the augmented matrix on the left by a special matrix.

Exchange and Permutation Matrices

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Multiplying any $4 \times n$ matrix on the left by the exchange matrix

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Notes on Matrix Multiplication

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However, mechanically, left multiplication corresponds to matrix vector multiplication on the columns.

$$MA \;=\; [Ma_{\bullet 1}\; Ma_{\bullet 2}\; \cdots \; Ma_{\bullet n}]$$

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$$AN = \begin{bmatrix} a_{1\bullet}N \\ a_{2\bullet}N \\ \vdots \\ a_{m\bullet}N \end{bmatrix}$$

Gaussian Elimination Matrices

The key step in Gaussian elimination is to transform a vector of the form

$$\begin{bmatrix} a \\ \alpha \\ b \end{bmatrix},$$

where $a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$, into one of the form

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This can be accomplished by left matrix multiplication as follows.

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$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

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Note that a Gaussian elimination matrix and its inverse are both lower triangular matrices.

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- if $M \in S$ is invertible, then $M^{-1} \in S$.

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Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

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$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

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Math 407 Begins!

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- The function to be minimized or maximized is called the *objective function*.
- The set of alternatives is called the feasible region (or constraint region).
- In this course, the feasible region is always taken to be a subset of $\mathbb{R}^n$ (real n-dimensional space) and the objective function is a function from $\mathbb{R}^n$ to $\mathbb{R}$.

What is linear programming (LP)?

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that are linear equality and/or linear inequality constraints.

- A linear function of the variables $x_1, x_2, \ldots, x_n$ is any function of the form
$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
for fixed $c_i \in \mathbb{R}$ $i = 1, \ldots, n$.

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- A linear equality constraint is any equation of the form

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- A linear equality constraint is any inequality of the form
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq \alpha,$$
or
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq \alpha,$$
where $\alpha, a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Compact Representation

$$
\begin{array}{lll}
\text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n & \\
\text{subject to} & a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq \alpha_i & i = 1, \ldots, s \\
& b_{i1}x_i + b_{i2}x_2 + \cdots + b_{in}x_n = \beta_i & i = 1, \ldots, r.
\end{array}
$$

Vector Inequalities: Componentwise

Let $x, y \in \mathbb{R}^n$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

We write $x \leq y$ if and only if

$$x_i \leq y_i, \ \ i = 1, 2, \ldots, n \ .$$

Matrix Notation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = c^Tx$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix Notation

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq \alpha_i \quad i = 1, \ldots, s$$

$$\Longleftrightarrow$$

$$Ax \leq a$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \ldots & a_{sn} \end{bmatrix} \qquad a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Matrix Notation

$$b_{i1}x_i + b_{i2}x_2 + \cdots + b_{in}x_n = \beta_i \quad i = 1, \ldots, r$$

$$\Longleftrightarrow$$

$$Bx = b$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \ldots & b_{rn} \end{bmatrix} \qquad b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_s \end{bmatrix}$$

LP's Matrix Notation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le a \text{ and } Bx = b \end{array}$$

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \ a = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \ b = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_s \end{bmatrix}.$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \\ a_{s1} & \dots & a_{sn} \end{bmatrix}, \ B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ & \ddots & \\ b_{r1} & \dots & b_{rn} \end{bmatrix}$$

Applications of Linear Programing

A short list:

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