Linear Programming

Lecture 2: Introduction to Linear Programming [1]

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Math 407: Introduction

What is linear programming?

Applications of Linear Programing

Example: Plastic Cup Factory

Introduction to LP Modeling

Graphical Solution of 2D LPs

Introduction to Sensitivity Analysis

The Theory of Linear Economic Models
- Production Models
- The Optimal Value Function and Marginal Values
- Duality: The *Hidden Hand* of the Market Place

LP Duality
- The Weak Duality Theorem of Linear Programming

What is optimization?

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A mathematical optimization problem is one in which some function is either maximized or minimized relative to a given set of alternatives.

- The function to be minimized or maximized is called the *objective function*.
- The set of alternatives is called the feasible region (or constraint region).
- In this course, the feasible region is always taken to be a subset of $\mathbb{R}^n$ (real n-dimensional space) and the objective function is a function from $\mathbb{R}^n$ to $\mathbb{R}$.

What is linear programming (LP)?

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A *linear program* is an optimization problem in finitely many variables having a linear objective function and a constraint region determined by a finite number of linear equality and/or inequality constraints.

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A *linear program* is an <u>optimization problem</u>

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that are linear equality and/or linear inequality constraints.

- A linear function of the variables $x_1, x_2, \ldots, x_n$ is any function f of the form

$$f(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

for fixed $c_i \in \mathbb{R}$ $i = 1, \ldots, n$.

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- A linear equality constraint is any equation of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \alpha,$$

  where $\alpha, a_1, a_2, \ldots, a_n \in \mathbb{R}$.

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- A linear inequality constraint is any inequality of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq \alpha,$$

or

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq \alpha,$$

where $\alpha, a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Compact Representation

$$
\begin{aligned}
\text{minimize} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} \quad & a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq \alpha_i \qquad i = 1, \ldots, s \\
& b_{i1}x_i + b_{i2}x_2 + \cdots + b_{in}x_n = \beta_i \qquad i = 1, \ldots, r.
\end{aligned}
$$

Vector Inequalities: Componentwise

Let $x, y \in \mathbb{R}^n$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

We write $x \leq y$ if and only if

$$x_i \leq y_i, \;\; i = 1, 2, \ldots, n \; .$$

Matrix Notation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = c^T x$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix Notation

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq \alpha_i \quad i = 1, \ldots, s$$

$$\Longleftrightarrow$$

$$Ax \leq a$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \ldots & a_{sn} \end{bmatrix} \qquad a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_s \end{bmatrix}$$

Matrix Notation

$$b_{i1}x_i + b_{i2}x_2 + \cdots + b_{in}x_n = \beta_i \quad i = 1, \ldots, r$$

$$\Longleftrightarrow$$

$$Bx = b$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \ldots & b_{rn} \end{bmatrix} \qquad b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_r \end{bmatrix}$$

LP's Matrix Notation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq a \text{ and } Bx = b \end{array}$$

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Applications of Linear Programing

A very short list:

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Example: Plastic Cup Factory

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is \$25 while the profit on a case of champagne glasses is \$20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

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Model this problem as an LP.

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4. Determine the *implicit constraints* and write them as a linear equation/inequality in the decision variables.

Decision Variables

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B = number of cases of beer mugs produced daily
C = number of cases of champagne glasses produced daily

Objective Function

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Maximize Profit: Profit = Revenue − Costs

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Maximize Profit: $\text{Profit} = \text{Revenue} - \text{Costs}$

$$\text{Profit} = 25B + 20C$$

Explicit Constraints

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Resin: $20B + 12C \leq 1800$

Labor: $B/15 + C/15 \leq 8$

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Implicit Constraints:
The decision variables are non-negative: $0 \leq B,\ 0 \leq C$

The Plastic Cup Factory LP Model

$$
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\text{maximize} \quad & 25B + 20C \\
\text{subject to} \quad & 20B + 12C \leq 1800 \\
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The Hardest Part of Modeling: Decision Variables

Once again, the first step in the modeling process, identification of the decision variables, is always the most difficult.

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Never be afraid to add more decision variables either to clarify the model or to improve its flexibility. Modern LP software easily solves problems with thousands of variables on a laptop, tens of thousands of variables on a server, or even tens of millions of variables on specialized hardware and networks. It is more important to get a correct, easily interpretable, and flexible model then to provide a compact minimalist model.

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LP model solutions found in many texts fall into the trap of trying to provide the most compact minimalist model with the fewest possible number of variables and constraints.
Do not repeat this error in developing your own models.

Graphical Solution of 2D LPs

We now graphically solve the LP model for the Plastic Cup Factory problem.

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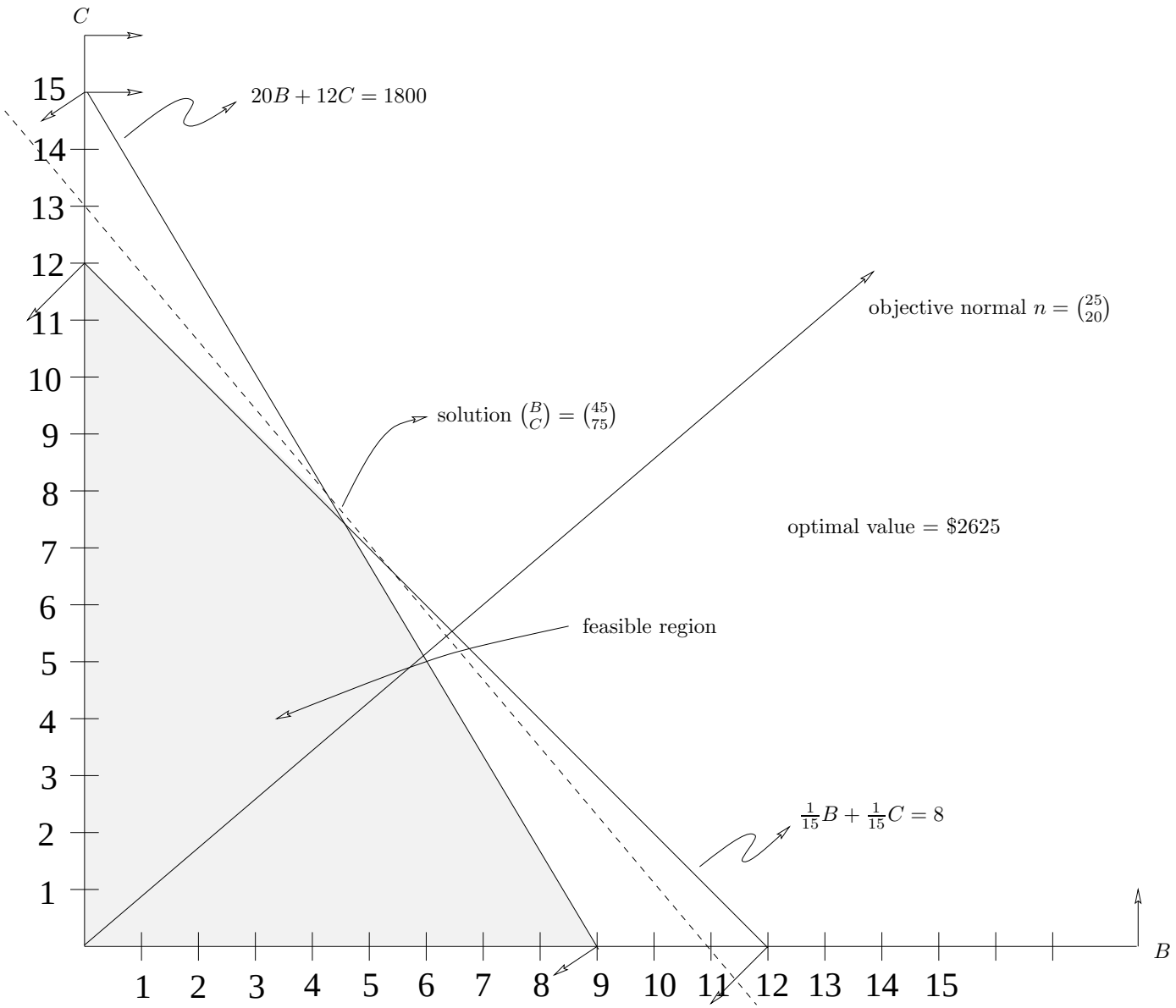
Graphical Solution of 2D LPs



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Step 1: Graph each of the linear constraints indicating on which side of the constraint the feasible region must lie with an arrow. Don't forget the implicit constraints!

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Step 3: Draw the gradient vector of the objective function.

Recap: Graphical Solution of 2D LPs

Step 1: Graph each of the linear constraints indicating on which side of the constraint the feasible region must lie with an arrow. Don't forget the implicit constraints!

Step 2: Shade in the feasible region.

Step 3: Draw the gradient vector of the objective function.

Step 4: Place a straight-edge perpendicular to the gradient vector.

Recap: Graphical Solution of 2D LPs

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Step 5: The set of points of intersection between the straight-edge and the feasible region is the set of solutions to the LP. Compute these points precisely along with the associated optimal value.

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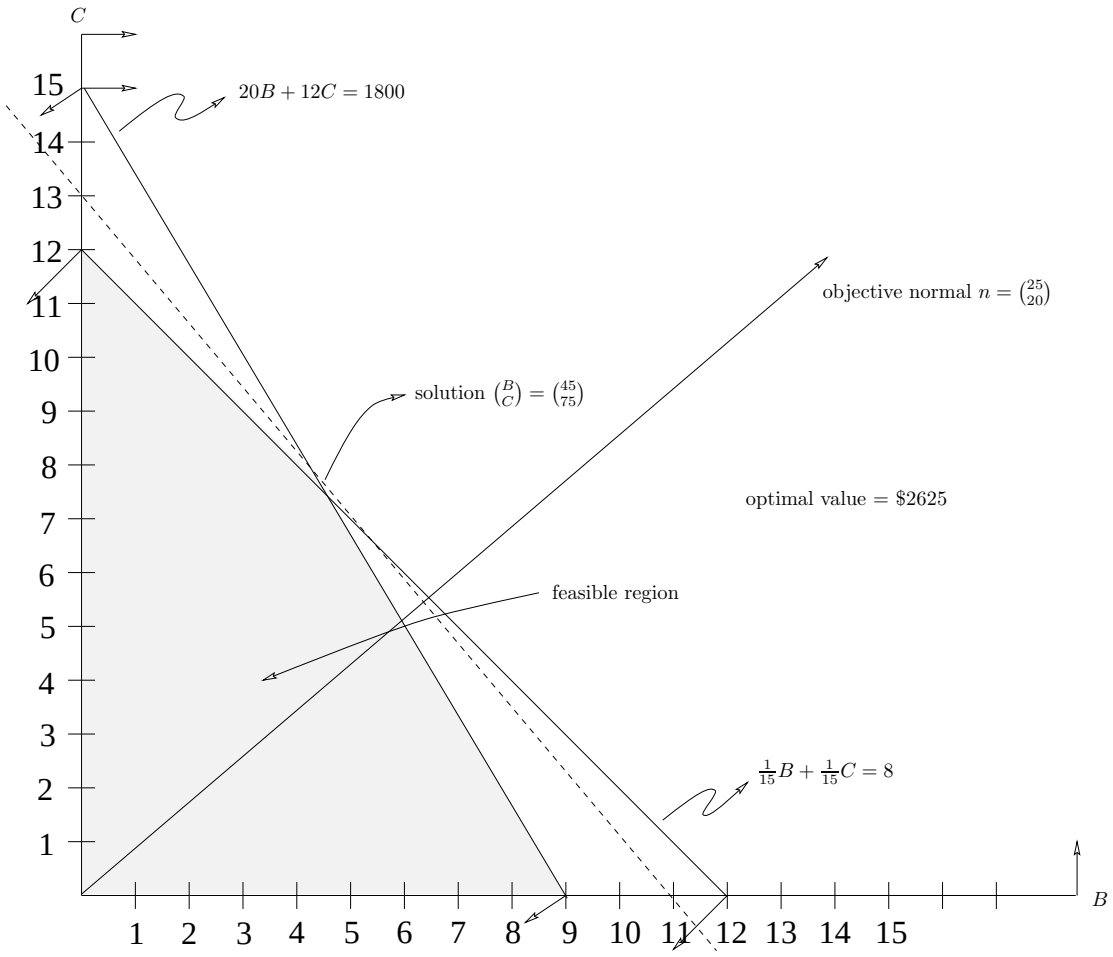
Vertex Solutions



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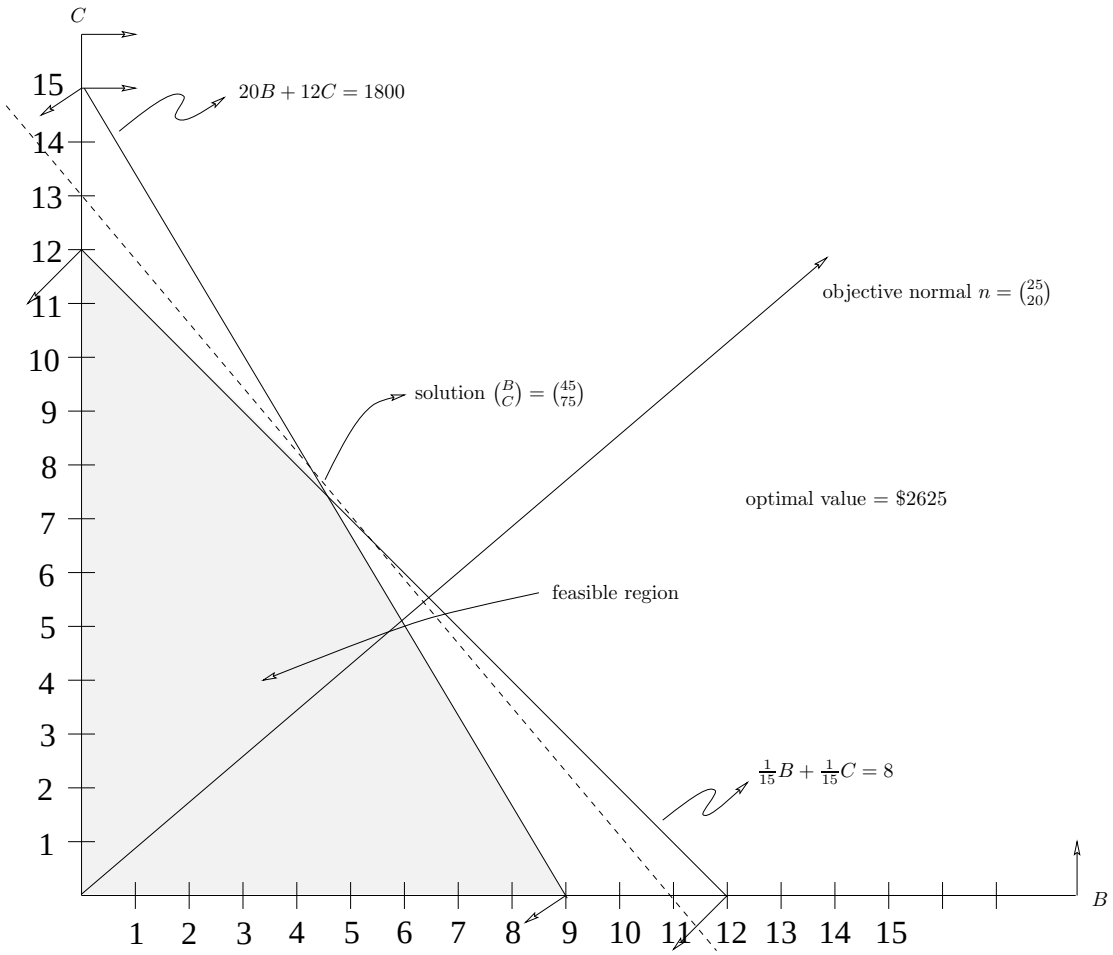

The optimal solution lies at a "corner point" or "vertex" of the feasible region.

Vertex Solutions

Conjecture: For a small range of perturbations to the resources, the vertex associated with the current optimal solution moves but remains optimal.

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It can be verified by direct computation that this indeed yields the optimal solution for small values of ϵ_1 and ϵ_2.

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The components of the gradient are called the marginal values for the resources.

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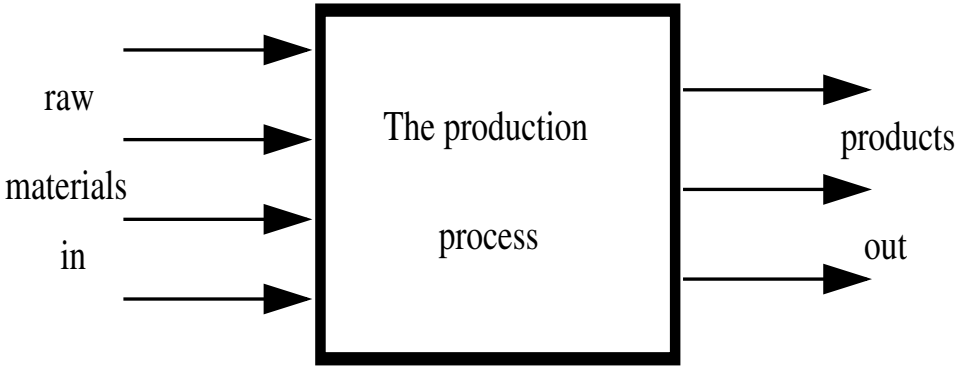
The Production Model



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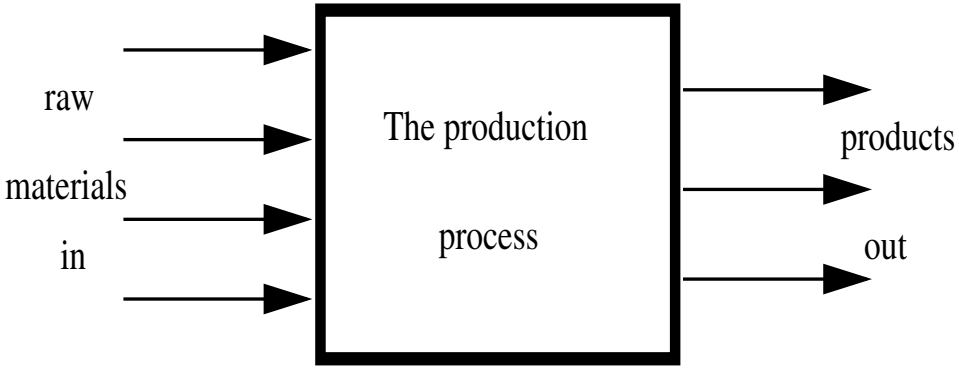

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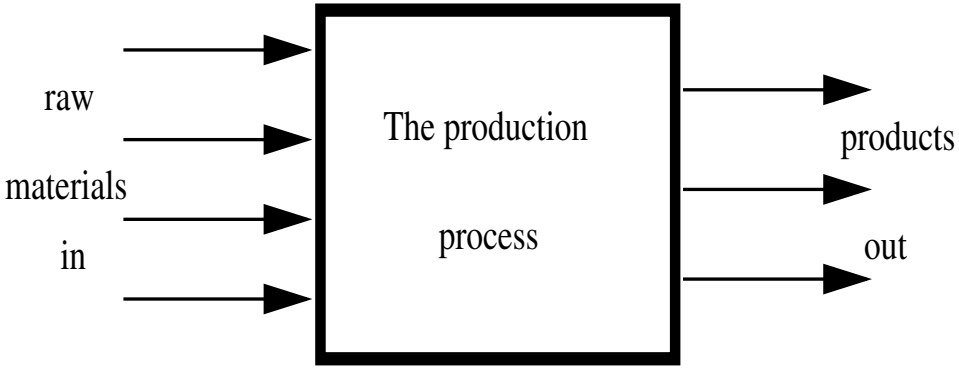

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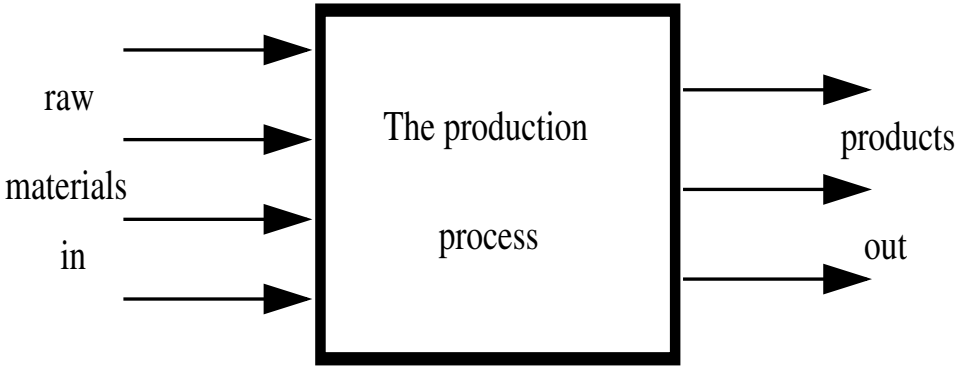

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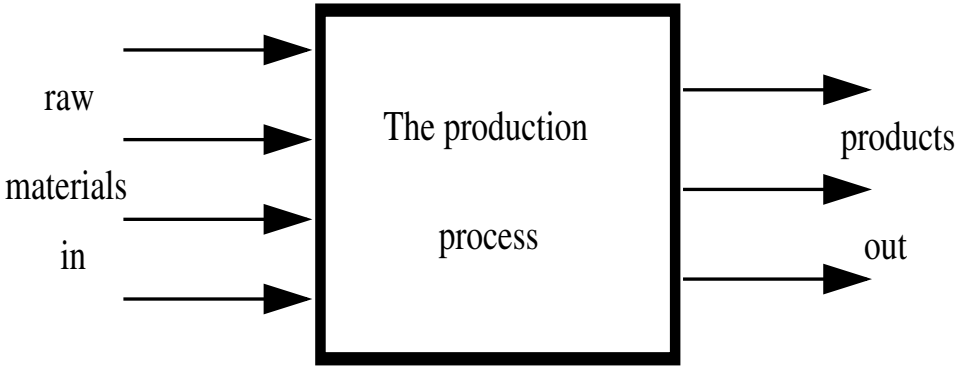

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Solution: The marginal values!

$$
\nabla v(\epsilon_1, \epsilon_2) = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}
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How can we model this mathematically?

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A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champagne glasses. The profit on a case of beer mugs is \$25 while the profit on a case of champagne glasses is \$20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champagne glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

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These price increases should wipe out the per unit profitability for cases of both beer mugs and champagne glasses.

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Let us compare this LP with the original LP.

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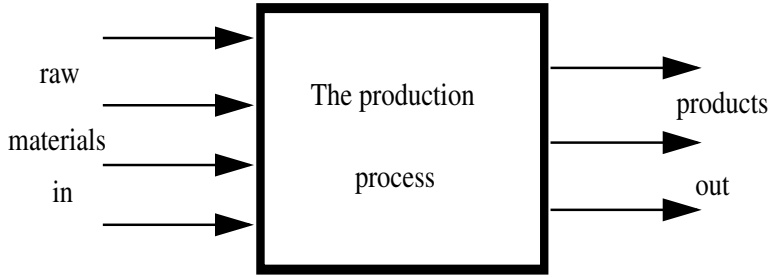

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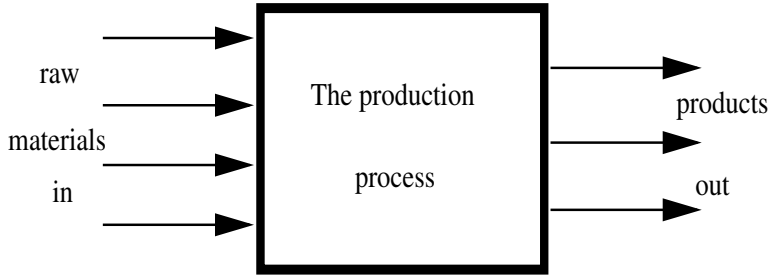

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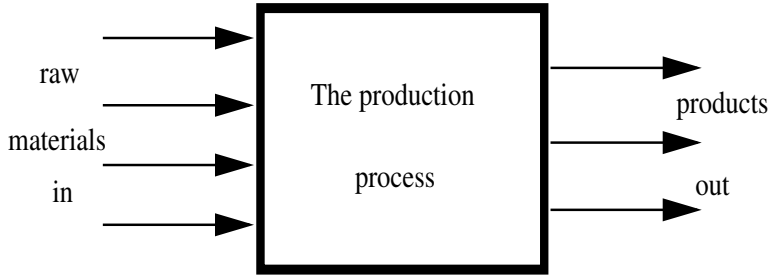

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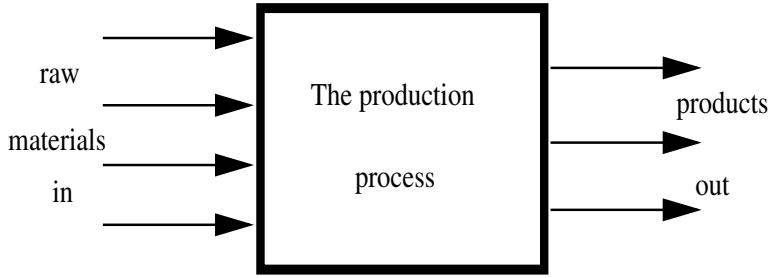
Recall the goal of the Market:

The market wants to make the most money possible from its resources by setting the highest prices it can without driving the producers out of business.



The marginal values give the per unit increase in the value of the resources due to the production process.

The marginal values should be the solution to the dual!
And indeed, they are the solution!

Linear Programming Duality: Matrix Notation

	$\mathcal{P}$			$\mathcal{D}$	
Primal:	max	$c^T x$	**Dual:**	min	$b^T y$
	s.t.	$Ax \leq b$		s.t.	$A^T y \geq c$
		$0 \leq x$			$0 \leq y$

The Weak Duality Theorem of Linear Programming

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Moreover, if $c^T \bar{x} = b^T \bar{y}$ with $\bar{x}$ feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$, then $\bar{x}$ must solve $\mathcal{P}$ and $\bar{y}$ must solve $\mathcal{D}$.

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\begin{aligned}
c^T x &= \sum_{j=1}^{n} c_j x_j \\
&\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j \quad \left[0 \leq x_j,\ c_j \leq \sum_{i=1}^{m} a_{ij} y_i \ \Rightarrow\ c_j x_j \leq \left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j\right]
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&= b^T y
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Test the WDT on the Plastic Cup Factory

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$$\text{Optimal Solution} = \begin{bmatrix} 45 \\ 75 \end{bmatrix}$$

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Equivalence of primal-dual objectives (WDT):

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$$c^T x = 25 \cdot 45 + 20 \cdot 75 = 2625$$

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Equivalence of primal-dual objectives (WDT):

$$c^T x = 25 \cdot 45 + 20 \cdot 75 = 2625 = 1800 \cdot \frac{5}{8} + 8 \cdot \frac{375}{2} = b^T y$$

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Example:

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & - & x_2 & & \\ & x_1 & - & x_2 & \leq & 1 \\ & -x_1 & + & x_2 & \leq & -2 \\ & 0 & \leq & x_1, & x_2 & \end{array}$$