

# Linear Programming

## Lecture 13: Sensitivity Analysis

Sensitivity Analysis

Silicon Chip Corporation

Break-even Prices and Reduced Costs

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Resource Variations, Marginal Values, and Range Analysis

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The Fundamental Theorem on Sensitivity Analysis

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We begin our study of sensitivity analysis with a concrete toy example.

# SILICON CHIP CORPORATION

A Silicon Valley firm specializes in making four types of silicon chips for personal computers. Each chip must go through four stages of processing before completion. First the basic silicon wafers are manufactured, second the wafers are laser etched with a micro circuit, next the circuit is laminated onto the chip, and finally the chip is tested and packaged for shipping. The production manager desires to maximize profits during the next month. During the next 30 days she has enough raw material to produce 4000 silicon wafers. Moreover, she has 600 hours of etching time, 900 hours of lamination time, and 700 hours of testing time. Taking into account depreciated capital investment, maintenance costs, and the cost of labor, each raw silicon wafer is worth \$1, each hour of etching time costs \$40, each hour of lamination time costs \$60, and each hour of inspection time costs \$10.

The production manager has formulated her problem as a profit maximization

# SILICON CHIP CORPORATION

Initial Tableau:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
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$x_1, x_2, x_3, x_4$  represent the number of 100 chip batches of the four types of chips.

The objective row coefficients correspond to dollars profit per 100 chip batch.

# SILICON CHIP CORPORATION

Optimal tableau:

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The optimal production schedule is

$$(x_1, x_2, x_3, x_4) = (0, 25, 10, 5),$$

and the optimal value is \$145,000.

## Break-even Prices and Reduced Costs

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At what sale price is it efficient to produce type 1 chip?

That is, what is the sale price  $p$  below which type 1 chip does not appear in the optimal production mix, and above which it does appear in the optimal mix?

This is called the **breakeven sale price** of type 1 chip.



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$$\text{lamination cost} = \text{no. hours} \times \text{cost per hour} = 20 \times 60 = 1200$$

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The cost per batch of 100 type 1 chips is \$1900.

The current sale price of each batch of 100 type 1 chips is

$\$2000 + \$1900 = \$3900$ , or equivalently, \$39 per chip.

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Let  $\theta$  denote the increase in sale price of type 1 chip needed for it to enter the optimal production mix.



## Break-even Prices and Reduced Costs

With this change to the sale price of type 1 chip the initial tableau for the LP becomes

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Suppose we repeat on this tableau all of the pivots that led to the previously optimal tableau.

What will the new tableau look like? That is, how does  $\theta$  appear in this new tableau?

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The nonsingular matrix  $R$  is called the *record matrix*.

## Break-even Prices and Reduced Costs

The optimal tableau has the form

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} RA & R & Rb \\ (c - A^T y)^T & -y^T & -b^T y \end{bmatrix}, \quad (4.1)$$

where  $0 \leq y$ ,  $A^T y \geq c$ , and the optimal value is  $b^T y$ .



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The corresponding new initial tableau is

$$\begin{bmatrix} A & I & b \\ (c + \Delta c)^T & 0 & 0 \end{bmatrix} . \quad (4.2)$$

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That is, we just add  $\Delta c$  to the objective row in the old optimal tableau.

## Break-even Prices and Reduced Costs

$$T = \left[ \begin{array}{ccc} RA & R & Rb \\ \Delta c^T + (c - A^T y)^T & -y^T & -b^T y \end{array} \right]$$

If  $T$  is a tableau, then it remains optimal if and only if

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or equivalently,

$$\Delta c \leq -(c - A^T y) .$$

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These inequalities place restrictions on how large the entries of  $\Delta c$  can be before one must pivot to obtain the new optimal tableau.



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$$c = \begin{pmatrix} 2000 \\ 3000 \\ 5000 \\ 4000 \end{pmatrix}, \quad \Delta c = \begin{pmatrix} \theta \\ 0 \\ 0 \\ 0 \end{pmatrix} = \theta e_1, \quad c + \Delta c = \begin{pmatrix} 2000 + \theta \\ 3000 \\ 5000 \\ 4000 \end{pmatrix} = c + \theta e_1$$

## Break-even Prices and Reduced Costs

$$\begin{bmatrix} RA & R & Rb \\ (c - A^T y)^T & -y^T & -b^T y \end{bmatrix} =$$

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$$\Delta c + (c - A^T y) = \theta e_1 + (c - A^T y) = \begin{pmatrix} \theta \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1500 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \theta - 1500 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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Thus, to preserve optimality we need  $\theta \leq 1500$ .

## Break-even Prices and Reduced Costs

That is, as soon as  $\theta$  increases beyond 1500, type 1 chip enters the optimal production mix, and for  $\theta = 1500$  we obtain multiple optimal solutions where type 1 chip may be in the optimal production mix if we so choose.



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For non-basic variables the break-even sale price can be read off from the reduced costs from the optimal tableau.

break-even price = current price + reduced cost =  $\$39 + \$15 = \$54$  .

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Recall that the objective row coefficients in the optimal tableau correspond to the following expression for the objective variable  $z$ :

$$z = 145000 - 1500x_1 - 5x_5 - 100x_7 - 50x_8.$$



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Recall that the objective row coefficients in the optimal tableau correspond to the following expression for the objective variable  $z$ :

$$z = 145000 - 1500x_1 - 5x_5 - 100x_7 - 50x_8.$$

Hence, if we make one batch of type 1 chip, we reduce our optimal value by \$1500. Thus, to recoup this loss we must charge \$1500 more for these chips yielding a break-even sale price of  $\$39 + \$15 = \$54$  per chip.

# Range Analysis for Objective Coefficients

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We now examine objective coefficient variations.

## Range Analysis for Objective Coefficients

Recall that to compute a breakeven price one needs to determine the change in the associated objective coefficient that make it efficient to introduce this activity into the optimal production mix, or equivalently, to determine the smallest change in the objective coefficient of this currently non-basic decision variable that requires one to bring it into the basis in order to maintain optimality.

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A related question is *what is the range of variation of a given objective coefficient that preserves the current basis as optimal?*

The answer to this question is an interval, possibly unbounded, on the real line within which a given objective coefficient can vary but these variations do not effect the currently optimal basis.

# SILICON CHIP CORPORATION

Initial Tableau	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

Opt. Tableau	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
	0.5	1	0	0	.015	0	0	-.05	25
	-5	0	0	0	-.05	1	0	-.5	50
	0	0	1	0	-.02	0	.1	0	10
	0.5	0	0	1	.015	0	-.1	.05	5
	-1500	0	0	0	-5	0	-100	-50	-145,000

## Range Analysis for Objective Coefficients

Silicon Chip Corp optimal tableau.

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
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-1500	0	0	0	-5	0	-100	-50	-145,000

In the Silicon Chip Corp problem the decision variable  $x_3$  associated with type 3 chips is in the optimal basis.

For what range of variations in  $c_3 = 5000$  does the current optimal basis  $\{x_2, x_3, x_4, x_6\}$  remain optimal?

## Range Analysis for Objective Coefficients

To answer this question we perturb the objective coef. of type 3 chip and write  $c_3 = 5000 + \theta$ .

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$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
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-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
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-1500	0	$\theta$	0	-5	0	-100	-50	-145,000

This is no-longer a simplex tableau. To recover a tableau we must pivot on the  $x_3$  column.

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
-1500	0	$\theta$	0	-5	0	-100	-50	-145,000

To recover a proper simplex tableau we must eliminate  $\theta$  from the objective row entry under  $x_3$ .

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
-1500	0	$\theta$	0	-5	0	-100	-50	-145,000

To recover a proper simplex tableau we must eliminate  $\theta$  from the objective row entry under  $x_3$ .

Multiply the 3<sup>rd</sup> row by  $-\theta$  and add it to the objective row to eliminate  $\theta$ .

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
-1500	0	0	0	$-5 + 0.02\theta$	0	$-100 - 0.1\theta$	-50	$-145,000 - 10\theta$

To remain optimal the objective row must remain non-positive.

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
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To remain optimal the objective row must remain non-positive.

$$\begin{aligned}
 -5 + 0.02\theta &\leq 0, & \text{or equivalently, } & \theta \leq 250 \\
 -100 - 0.1\theta &\leq 0, & \text{or equivalently, } & -1000 \leq \theta
 \end{aligned}$$



$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
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which implies

$$4000 \leq c_3 \leq 5250.$$

since originally  $c_3 = 5000$ .

What is the range of the objective coefficient for type 4 chips that preserves the current basis as optimal?

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$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
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0.5	1	0	0	.015	0	0	-.05	25
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0.5	0	0	1	.015	0	-.1	.05	5
-1500 - 0.5 $\theta$	0	0	0	-5 - 0.015 $\theta$	0	-100 + 0.1 $\theta$	-50 - 0.05 $\theta$	-145,000 - 5 $\theta$

To preserve dual feasibility we must have

$$\begin{aligned}
 -1500 - 0.5\theta &\leq 0, & \text{or equivalently,} & & -3000 &\leq \theta \\
 -5 - 0.015\theta &\leq 0, & \text{or equivalently,} & & -333.\bar{3} &\leq \theta \\
 -100 + 0.1\theta &\leq 0, & \text{or equivalently,} & & \theta &\leq 1000 \\
 -50 - 0.05\theta &\leq 0, & \text{or equivalently,} & & -1000 &\leq \theta
 \end{aligned}$$

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 -50 - 0.05\theta &\leq 0, & \text{or equivalently,} & & -1000 &\leq \theta
 \end{aligned}$$

Thus,

$$-333.\bar{3} \leq \theta \leq 1000,$$

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 \end{aligned}$$

Thus,

$$-333.\bar{3} \leq \theta \leq 1000,$$

and the range for  $c_4$  is

$$3666.\bar{6} \leq c_4 \leq 5000$$

since originally  $c_4 = 4000$ .



What is the range for the objective coefficient for  $x_2$ ?

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
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0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
-1500	0	0	0	-5	0	-100	-50	-145,000

$$-333.\bar{3} \leq \theta \leq 1000$$

and

$$1666.\bar{6} \leq c_2 \leq 3000$$

# Resource Variations, Marginal Values, and Range Analysis

We now consider questions concerning the effect of resource variations on the optimal solution.

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Suppose we wish to purchase more silicon wafers this month. Before doing so, we need to answer three obvious questions.

- ▶ How many should we purchase?
- ▶ What is the most that we should pay for them?
- ▶ After the purchase, what is the new optimal production schedule?



The technique we develop for answering these questions is similar to the technique used to determine objective coefficient ranges.

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We begin by introducing a variable  $\theta$  for the number of silicon wafers that will be purchased, and then determine how this variable appears in the tableau after using the same simplex pivots encoded in the matrix  $G$  given above.

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	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
raw wafers	100	100	100	100	1	0	0	0	$4000 + \theta$
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
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## General RHS Perturbations

To effect the same simplex pivots we multiply the perturbed initial tableau by the elimination matrix  $G$ .

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This tableau is optimal if it is primal feasible.



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The new tableau is dual feasible.

This tableau is optimal if it is primal feasible.

That is, the new tableau is optimal as long as

$$0 \leq Rb + R\Delta b \Leftrightarrow -Rb \leq R\Delta b .$$

## General RHS Perturbations

$$\left[ \begin{array}{cc|c} RA & R & Rb \\ \hline (c - A^T y)^T & -y^T & -y^T b \end{array} \right] =$$

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
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$$R = \begin{bmatrix} .015 & 0 & 0 & -.05 \\ -.05 & 1 & 0 & -.5 \\ -.02 & 0 & .1 & 0 \\ .015 & 0 & -.1 & .05 \end{bmatrix}$$

## General RHS Perturbations

$$\left[ \begin{array}{cc|c} RA & R & Rb \\ \hline (c - A^T y)^T & -y^T & -y^T b \end{array} \right] =$$

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-1500	0	0	0	-5	0	-100	-50	-145,000

$$R = \begin{bmatrix} .015 & 0 & 0 & -.05 \\ -.05 & 1 & 0 & -.5 \\ -.02 & 0 & .1 & 0 \\ .015 & 0 & -.1 & .05 \end{bmatrix} \quad y = \begin{pmatrix} 5 \\ 0 \\ 100 \\ 50 \end{pmatrix}$$

## General RHS Perturbations

$$\left[ \begin{array}{cc|c} RA & R & Rb \\ \hline (c - A^T y)^T & -y^T & -y^T b \end{array} \right] =$$

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
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<hr/>								
-1500	0	0	0	-5	0	-100	-50	-145,000

$$R = \begin{bmatrix} .015 & 0 & 0 & -.05 \\ -.05 & 1 & 0 & -.5 \\ -.02 & 0 & .1 & 0 \\ .015 & 0 & -.1 & .05 \end{bmatrix} \quad y = \begin{pmatrix} 5 \\ 0 \\ 100 \\ 50 \end{pmatrix} \quad Rb = \begin{pmatrix} 25 \\ 50 \\ 10 \\ 5 \end{pmatrix}$$

## General RHS Perturbations

How do variations the raw wafer resource effect the optimal tableau?

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
raw wafers	100	100	100	100	1	0	0	0	$4000 + \theta$
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

$$b + \Delta b = b + \theta e_1 = \begin{pmatrix} 4000 \\ 600 \\ 900 \\ 700 \end{pmatrix} + \theta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

## General RHS Perturbations

$$\left[ \begin{array}{cc|c} RA & R & Rb + R\Delta b \\ \hline (c - A^T y)^T & -y^T & -y^T b - y^T \Delta b \end{array} \right] =$$

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$	
0.5	1	0	0	.015	0	0	-.05	25	
-5	0	0	0	-.05	1	0	-.5	50	
0	0	1	0	-.02	0	.1	0	10	$+R\Delta b$
0.5	0	0	1	.015	0	-.1	.05	5	
-1500	0	0	0	-5	0	-100	-50	-145,000	$-y^T \Delta b$

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0.5	1	0	0	.015	0	0	-.05	25	
-5	0	0	0	-.05	1	0	-.5	50	
0	0	1	0	-.02	0	.1	0	10	$+R\Delta b$
0.5	0	0	1	.015	0	-.1	.05	5	
-1500	0	0	0	-5	0	-100	-50	-145,000	$-y^T \Delta b$

$$Rb + \theta Re_1 = \begin{pmatrix} 25 \\ 50 \\ 10 \\ 5 \end{pmatrix} + \theta \begin{pmatrix} 0.015 \\ -0.05 \\ -0.02 \\ 0.015 \end{pmatrix} = \begin{pmatrix} 25 + \theta 0.015 \\ 50 - \theta 0.05 \\ 10 - \theta 0.02 \\ 5 + \theta 0.015 \end{pmatrix}$$



## General RHS Perturbations

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-5	0	0	0	-.05	1	0	-.5	50	
0	0	1	0	-.02	0	.1	0	10	$+R\Delta b$
0.5	0	0	1	.015	0	-.1	.05	5	
-1500	0	0	0	-5	0	-100	-50	-145,000	$-y^T \Delta b$

$$0 \leq Rb + \theta Re_1 = \begin{pmatrix} 25 \\ 50 \\ 10 \\ 5 \end{pmatrix} + \theta \begin{pmatrix} 0.015 \\ -0.05 \\ -0.02 \\ 0.015 \end{pmatrix} = \begin{pmatrix} 25 + \theta 0.015 \\ 50 - \theta 0.05 \\ 10 - \theta 0.02 \\ 5 + \theta 0.015 \end{pmatrix}$$

## General RHS Perturbations

To preserve primal feasibility we need  $-Rb \leq \theta R\Delta b$ , i.e.

$$-\begin{pmatrix} 25 \\ 50 \\ 10 \\ 5 \end{pmatrix} \leq \theta \begin{pmatrix} 0.015 \\ -0.05 \\ -0.02 \\ 0.015 \end{pmatrix},$$

or equivalently,

$$\begin{array}{rcll} -25 & \leq & .015\theta & \text{implies } \theta \geq -5000/3 \\ -50 & \leq & -.05\theta & \text{implies } \theta \leq 1000 \\ -10 & \leq & -.02\theta & \text{implies } \theta \leq 500 \\ -5 & \leq & .015\theta & \text{implies } \theta \geq -1000/3 \end{array}$$

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This reduces to the simple inequality

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$$-5 \leq .015\theta \quad \text{implies} \quad \theta \geq -1000/3$$

This reduces to the simple inequality

$$-\frac{1000}{3} \leq \theta \leq 500.$$

The interval  $3666.\bar{6} \leq b_1 \leq 4500$  is called the *range of the raw chip resource in the optimal solution*.

## General RHS Perturbations

If  $-\frac{1000}{3} \leq \theta \leq 500$ , then the optimal solution is given by

$$\begin{pmatrix} x_2 \\ x_6 \\ x_3 \\ x_4 \end{pmatrix} = Rb + R\Delta b = \begin{pmatrix} 25 + .015\theta \\ 50 - .05\theta \\ 10 - .02\theta \\ 5 + .015\theta \end{pmatrix}$$

with optimal value

$$y^T b + y^T \Delta b = 145000 + 5\theta.$$

## General RHS Perturbations

Now examine the profit expression

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The marginal value is the per unit increased value of this resource due to the production process.

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The dual value 5 is called the *shadow price*, or *marginal value*, for the raw silicon wafer resource.

The marginal value is the per unit increased value of this resource due to the production process.

Since we currently pay \$1 per wafer. If another vendor sells them at \$2.50 per wafer, then we should buy them since our unit increase in profit with this purchase price is  $\$5 - \$1.5 = \$3.5$  since \$2.5 is \$1.5 greater than the \$1 we now pay.

Thus we should purchase 500 raw wafers at a purchase price of no more than  $\$5 + \$1 = \$6$  dollars per wafer.

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The new optimal production schedule is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 25 + .015\theta \\ 10 - .02\theta \\ 5 + .015\theta \end{pmatrix}_{\theta=500} = \begin{pmatrix} 0 \\ 32.5 \\ 0 \\ 12.5 \end{pmatrix} .$$

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Should we purchase more than 500 chips?

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	$25 + .015\theta$
-5	0	0	0	-.05	1	0	-.5	$50 - .05\theta$
0	0	1	0	-.02	0	.1	0	$10 - .02\theta$
0.5	0	0	1	.015	0	-.1	.05	$5 + .015\theta$
-1500	0	0	0	-5	0	-100	-50	$-145,000 - 5\theta$

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	$25 + .015\theta$
-5	0	0	0	-.05	1	0	-.5	$50 - .05\theta$
0	0	1	0	-.02	0	.1	0	$10 - .02\theta$ ←
0.5	0	0	1	.015	0	-.1	.05	$5 + .015\theta$
-1500	0	0	0	-5	0	-100	-50	$-145,000 - 5\theta$



$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	$25 + .015\theta$
-5	0	0	0	-.05	1	0	-.5	$50 - .05\theta$
0	0	1	0	-.02	0	.1	0	$10 - .02\theta$ ←
0.5	0	0	1	.015	0	-.1	.05	$5 + .015\theta$
-1500	0	0	0	-5	0	-100	-50	$-145,000 - 5\theta$

0.5	1	.75	0	0	0	.075	-.05	32.5
-5	0	-2.5	0	0	1	-.25	-.5	25
0	0	-50	0	1	0	-5	0	$-500 + \theta$
0.5	0	.75	1	0	0	-.025	.05	12.5
-1500	0	-250	0	0	0	-125	-50	-147500

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	$25 + .015\theta$
-5	0	0	0	-.05	1	0	-.5	$50 - .05\theta$
0	0	1	0	-.02	0	.1	0	$10 - .02\theta$ ←
0.5	0	0	1	.015	0	-.1	.05	$5 + .015\theta$
-1500	0	0	0	-5	0	-100	-50	$-145,000 - 5\theta$

0.5	1	.75	0	0	0	.075	-.05	32.5
-5	0	-2.5	0	0	1	-.25	-.5	25
0	0	-50	0	1	0	-5	0	$-500 + \theta$
0.5	0	.75	1	0	0	-.025	.05	12.5
-1500	0	-250	0	0	0	-125	-50	-147500

Do not purchase more than 500 since the wafer resource becomes slack.

## RHS Range Analysis: Etching Time

Let us now do a range analysis on the etching time resource  $b_2$ .

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	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

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raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	$600 + \theta$
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raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	$600 + \theta$
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

$$b + \Delta b = b + \theta e_2 = \begin{pmatrix} 4000 \\ 600 \\ 900 \\ 700 \end{pmatrix} + \theta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

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raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	$600 + \theta$
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
	2000	3000	5000	4000	0	0	0	0	0

$$b + \Delta b = b + \theta e_2 = \begin{pmatrix} 4000 \\ 600 \\ 900 \\ 700 \end{pmatrix} + \theta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The new rhs in the opt. tableau is  $Rb + \theta Re_2$  since  $\Delta b = \theta e_2$ .

## RHS Range Analysis: Etching Time

$$Rb + R\Delta b = \begin{pmatrix} 25 \\ 50 + \theta \\ 10 \\ 5 \end{pmatrix}$$



## RHS Range Analysis: Etching Time

$$0 \leq Rb + R\Delta b = \begin{pmatrix} 25 \\ 50 + \theta \\ 10 \\ 5 \end{pmatrix}$$

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To preserve primal feasibility we only require

$$0 \leq 50 + \theta,$$

or equivalently,

$$-50 \leq \theta.$$

## RHS Range Analysis: Etching Time

$$0 \leq Rb + R\Delta b = \begin{pmatrix} 25 \\ 50 + \theta \\ 10 \\ 5 \end{pmatrix}$$

To preserve primal feasibility we only require

$$0 \leq 50 + \theta,$$

or equivalently,

$$-50 \leq \theta.$$

Therefore, the range for  $b_2$  is

$$[550, +\infty) .$$

## RHS Range Analysis: Etching Time

What is the shadow price for etching, and what does it mean?

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
-1500	0	0	0	-5	0	-100	-50	-145,000

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$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
<hr/>								
-1500	0	0	0	-5	0	-100	-50	-145,000

The shadow price, or marginal value, is 0 since we have surplus etching time in the optimal solution.

## RHS Range Analysis: Etching Time

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$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
<hr/>								
-1500	0	0	0	-5	0	-100	-50	-145,000

The shadow price, or marginal value, is 0 since we have surplus etching time in the optimal solution.

Additional hours of etching time do not change current profit levels.

## RHS Range Analysis: Lamination Time

What is the range for lamination time, and what is its marginal value?

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
-1500	0	0	0	-5	0	-100	-50	-145,000

## RHS Range Analysis: Lamination Time

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
<hr/>								
-1500	0	0	0	-5	0	-100	-50	-145,000

$$0 \leq Rb + R\Delta b = \begin{pmatrix} 25 \\ 50 \\ 10 + 0.1\theta \\ 5 - 0.1\theta \end{pmatrix}$$



## RHS Range Analysis: Lamination Time

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
<hr/>								
-1500	0	0	0	-5	0	-100	-50	-145,000

$$0 \leq Rb + R\Delta b = \begin{pmatrix} 25 \\ 50 \\ 10 + 0.1\theta \\ 5 - 0.1\theta \end{pmatrix}$$

$$0 \leq 10 + 0.1\theta, \quad \text{or equivalently,} \quad -100 \leq \theta$$

$$0 \leq 5 - 0.1\theta, \quad \text{or equivalently,} \quad \theta \leq 50$$

## RHS Range Analysis: Lamination Time

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0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
<hr/>								
-1500	0	0	0	-5	0	-100	-50	-145,000

$$0 \leq Rb + R\Delta b = \begin{pmatrix} 25 \\ 50 \\ 10 + 0.1\theta \\ 5 - 0.1\theta \end{pmatrix}$$

$$\begin{aligned} 0 &\leq 10 + 0.1\theta, & \text{or equivalently,} & & -100 &\leq \theta \\ 0 &\leq 5 - 0.1\theta, & \text{or equivalently,} & & \theta &\leq 50 \end{aligned}$$

Therefore,

$$800 \leq b_3 \leq 950 .$$

## RHS Range Analysis: Lamination Time

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
-1500	0	0	0	-5	0	-100	-50	-145,000

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$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
<hr/>								
-1500	0	0	0	-5	0	-100	-50	-145,000

The shadow price, or marginal value, for lamination time is \$100.

## RHS Range Analysis: Lamination Time

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
-1500	0	0	0	-5	0	-100	-50	-145,000

The shadow price, or marginal value, for lamination time is \$100. Each additional hour of lamination time improves profitability by \$100.

## RHS Range Analysis: Lamination Time

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
-1500	0	0	0	-5	0	-100	-50	-145,000

The shadow price, or marginal value, for lamination time is \$100. Each additional hour of lamination time improves profitability by \$100.

If we are able to obtain 50 additional hours of lamination time this month, how much would we be willing to pay for it beyond what we currently pay?

## RHS Range Analysis: Lamination Time

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
-1500	0	0	0	-5	0	-100	-50	-145,000

The shadow price, or marginal value, for lamination time is \$100. Each additional hour of lamination time improves profitability by \$100.

If we are able to obtain 50 additional hours of lamination time this month, how much would we be willing to pay for it beyond what we currently pay?

**\$5,000**

# Pricing Out New Products

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(a) Is it efficient to produce?

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- (a) Is it efficient to produce?
- (b) If it is efficient to produce, what is the new optimal production schedule?

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Second, determine how the change to the initial tableau propagates through to the optimal tableau.

This propagation is determined by multiplying the new initial tableau through by the pivot matrix  $G$ .



## Pricing Out New Products

The initial tableau is altered by the addition of a new column:

$$\begin{bmatrix} a_{\text{new}} & A & I & b \\ c_{\text{new}} & c^T & 0 & 0 \end{bmatrix} \cdot$$

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Multiplying on the left by the matrix  $G$  gives

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} a_{\text{new}} & A & I & b \\ c_{\text{new}} & c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} Ra_{\text{new}} & RA & R & Rb \\ c_{\text{new}} - a_{\text{new}}^T y & (c - A^T y)^T & -y^T & -y^T b \end{bmatrix}.$$

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The expression  $(c_{\text{new}} - a_{\text{new}}^T y)$  determines whether this new tableau is optimal or not.

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The expression  $(c_{\text{new}} - a_{\text{new}}^T y)$  determines whether this new tableau is optimal or not.

If  $0 < (c_{\text{new}} - a_{\text{new}}^T y)$ , then the new tableau is not optimal. In this case the new product is efficient to produce.

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If  $(c_{\text{new}} - a_{\text{new}}^T y) > 0$ , then we say that the new product **does price out** and it should be introduced into the optimal production mix.

The new optimal production mix is found by applying the standard primal simplex algorithm to the tableau since this tableau is primal feasible but not dual feasible.



## Pricing Out New Products

Returning to the Silicon Chip Corp. problem, the new chip under consideration has

$$a_{\text{new}} = \begin{pmatrix} 100 \\ 10 \\ 10 \\ 10 \end{pmatrix} .$$

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The stated sale price or revenue for each 100 chip batch of the new chip is \$3310.

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We need to compute  $c_{\text{new}}$ .

The stated sale price or revenue for each 100 chip batch of the new chip is \$3310. We need to subtract from this number the cost of producing each 100 chip batch.

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We have that each raw silicon wafer is worth \$1, each hour of etching time costs \$40, each hour of lamination time costs \$60, and each hour of inspection time costs \$10.

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$$\begin{aligned} & 100 \times 1 \quad (\text{cost of the raw wafers}) \\ + & 10 \times 40 \quad (\text{cost of etching time}) \end{aligned}$$



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Hence the profit on each 100 chip batch of these new chips is  $\$3310 - \$1200 = \$2110$ , or \$21.10 per chip, and so

$$c_{\text{new}} = 2110.$$

## Pricing Out New Products

Pricing out the new chip gives

$$c_{\text{new}} - \mathbf{a}_{\text{new}}^T \mathbf{y} = 2110 - \begin{pmatrix} 100 \\ 10 \\ 10 \\ 10 \end{pmatrix}^T \begin{pmatrix} 5 \\ 0 \\ 100 \\ 50 \end{pmatrix} = 2110 - 2000 = 110 .$$

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The new chip prices out positive, and so it will be efficient to produce.

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The new chip prices out positive, and so it will be efficient to produce.

The new column in the tableau associated with this chip is

$$\begin{pmatrix} Ra_{\text{new}} \\ c_{\text{new}} - a_{\text{new}}^T y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 110 \end{pmatrix}.$$

## Pricing Out New Products

$x_{\text{new}}$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
1	0.5	1	0	0	.015	0	0	-.05	25
0	-5	0	0	0	-.05	1	0	-.5	50
-1	0	0	1	0	-.02	0	.1	0	10
1	0.5	0	0	1	.015	0	-.1	.05	5
110	-1500	0	0	0	-5	0	-100	-50	-145,000



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110	-1500	0	0	0	-5	0	-100	-50	-145,000
0	0	1	0	-1	0	0	.1	-.1	20
0	-5	0	0	0	-.05	1	0	-.5	50
0	.5	0	1	1	-.005	0	0	.05	15
1	0.5	0	0	1	.015	0	-.1	.05	5
0	-1555	0	0	-110	-6.65	0	-88.9	-55.5	-145550

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-1	0	0	1	0	-.02	0	.1	0	10
1	0.5	0	0	1	.015	0	-.1	.05	5
110	-1500	0	0	0	-5	0	-100	-50	-145,000
0	0	1	0	-1	0	0	.1	-.1	20
0	-5	0	0	0	-.05	1	0	-.5	50
0	.5	0	1	1	-.005	0	0	.05	15
1	0.5	0	0	1	.015	0	-.1	.05	5
0	-1555	0	0	-110	-6.65	0	-88.9	-55.5	-145550

The new optimal solution is  $(x_{\text{new}}, x_1, x_2, x_3, x_4) = (5, 0, 20, 15, 0)$ .

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0	0	1	0	-1	0	0	.1	-.1	20
0	-5	0	0	0	-.05	1	0	-.5	50
0	.5	0	1	1	-.005	0	0	.05	15
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The new optimal solution is  $(x_{\text{new}}, x_1, x_2, x_3, x_4) = (5, 0, 20, 15, 0)$ .

The new shadow prices are  $(y_1, y_2, y_3, y_4) = (6.65, 0, 89.9, 55.5)$ .

## Pricing Out New Products

Consider a different new chip.

This chip requires 15 hours each of etching and testing, and 30 hours of lamination time per 100 chip batch.

What is the breakeven sale price of this new chip?

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$b$
0.5	1	0	0	.015	0	0	-.05	25
-5	0	0	0	-.05	1	0	-.5	50
0	0	1	0	-.02	0	.1	0	10
0.5	0	0	1	.015	0	-.1	.05	5
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# Pricing Out New Products

Costs of production are

$$\text{costs} = \begin{pmatrix} 1 \\ 40 \\ 60 \\ 10 \end{pmatrix}^T \begin{pmatrix} 100 \\ 15 \\ 30 \\ 15 \end{pmatrix} = \$2650$$

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Breakeven sale price = \$2650 + \$4250 = \$6900. Or equivalently, \$69 per chip.

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We now wish to simultaneously determine if either or both of the new chips are efficient to produce.



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$$\begin{bmatrix} Ra_{\text{new1}} & Ra_{\text{new2}} & RA & R & Rb \\ c_{\text{new1}} - a_{\text{new1}}^T y & c_{\text{new2}} - a_{\text{new2}}^T y & (c - A^T y)^T & -y^T & -y^T b \end{bmatrix}$$

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Then pivot to optimality.

# The Fundamental Theorem on Sensitivity Analysis

Let  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ , and  $c \in \mathbb{R}^n$ .

$$\begin{array}{ll} \mathcal{P} & \text{maximize} \quad c^T x \\ & \text{subject to} \quad Ax \leq b, \quad 0 \leq x. \end{array}$$

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We associate to  $\mathcal{P}$  the *optimal value function*

$V : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\pm\infty\}$  defined by

$$V(u) = \begin{aligned} & \text{maximize} && c^T x \\ & \text{subject to} && Ax \leq b + u, \quad 0 \leq x \end{aligned}$$

for all  $u \in \mathbb{R}^m$ .

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$$\mathcal{F}(u) = \{x \in \mathbb{R}^n \mid Ax \leq b + u, \quad 0 \leq x\}$$

denote the feasible region for the LP associated with value  $V(u)$ .

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Let

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denote the feasible region for the LP associated with value  $V(u)$ .

If  $\mathcal{F}(u) = \emptyset$  for some  $u \in \mathbb{R}^m$ , we define  $V(u) = -\infty$ .



# The Fundamental Theorem on Sensitivity Analysis

*Theorem: If  $\mathcal{P}$  is primal nondegenerate, i.e. the optimal value is finite and no basic variable in any optimal tableau takes the value zero, then the dual solution  $y^*$  is unique and there is an  $\epsilon > 0$  such that*

$$V(u) = b^T y^* + u^T y^* \quad \text{whenever } |u_i| \leq \epsilon, \quad i = 1, \dots, m .$$

*Thus, in particular, the optimal value function  $V$  is differentiable at  $u = 0$  with  $\nabla V(0) = y^*$ .*

# The Fundamental Theorem on Sensitivity Analysis: Proof

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} RA & R & Rb \\ (c - A^T y^*)^T & -(y^*)^T & -b^T y^* \end{bmatrix}$$

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$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b + u \\ c^T & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b + u \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} RA & R & Rb + Ru \\ (c - A^T y^*)^T & -(y^*)^T & -b^T y^* - u^T y^* \end{bmatrix}$$

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Non-degeneracy implies that  $Rb > 0$  so there is an  $\epsilon > 0$  such that

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Non-degeneracy implies that  $Rb > 0$  so there is an  $\epsilon > 0$  such that

$$Rb > \epsilon \mathbf{1} .$$

By continuity, there is a  $\delta > 0$  such that

$$|(Ru)_i| \leq \epsilon \quad \text{whenever} \quad |u_i| \leq \delta \quad \forall i = 1, 2, \dots, n.$$

# The Fundamental Theorem on Sensitivity Analysis: Proof

$$\begin{bmatrix} RA & R & Rb + Ru \\ (c - A^T y^*)^T & -(y^*)^T & -b^T y^* - u^T y^* \end{bmatrix}$$

Non-degeneracy implies that  $Rb > 0$  so there is an  $\epsilon > 0$  such that

$$Rb > \epsilon \mathbf{1} .$$

By continuity, there is a  $\delta > 0$  such that

$$|(Ru)_i| \leq \epsilon \quad \text{whenever} \quad |u_i| \leq \delta \quad \forall i = 1, 2, \dots, n.$$

Hence  $Rb + Ru > 0$  whenever  $|u_i| \leq \delta \quad \forall i = 1, 2, \dots, n.$