Math 407A: Linear Optimization

Lecture 11: The Dual Simplex Algorithm

Math Dept, University of Washington

The Dual Simplex Algorithm

$$
\begin{array}{lll}
\mathcal{P} & \text{maximize} & -4x_1 - 2x_2 - x_3 \\
& \text{subject to} & -x_1 - x_2 + 2x_3 \le -3 \\
& & -4x_1 - 2x_2 + x_3 \le -4 \\
& & x_1 + x_2 - 4x_3 \le 2 \\
& & 0 \le x_1, x_2, x_3
\end{array}
$$

$$
\mathcal{P} \quad \begin{array}{lr} \text{maximize} & -4x_1 - 2x_2 - x_3 \\ \text{subject to} & -x_1 - x_2 + 2x_3 \le -3 \\ & -4x_1 - 2x_2 + x_3 \le -4 \\ & x_1 + x_2 - 4x_3 \le \ \ 2 \\ & 0 \le x_1, x_2, x_3 \end{array}
$$

$$
\mathcal{D} \quad \begin{array}{lr} \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ \text{subject to} & -y_1 - 4y_2 + y_3 \ge -4 \\ & -y_1 - 2y_2 + y_3 \ge -2 \\ & 2y_1 + y_2 - 4y_3 \ge -1 \\ & 0 \le y_1, y_2, y_3 \end{array}
$$

$$\mathcal{P} \quad \begin{array}{ll} \text{maximize} & -4x_1 - 2x_2 - x_3 \\ \text{subject to} & -x_1 - x_2 + 2x_3 \leq -3 \\ & -4x_1 - 2x_2 + x_3 \leq -4 \\ & x_1 + x_2 - 4x_3 \leq \ \ 2 \\ & 0 \leq x_1, x_2, x_3 \end{array}$$

$$\mathcal{D} \quad \begin{array}{ll} \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ \text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\ & -y_1 - 2y_2 + y_3 \geq -2 \\ & 2y_1 + y_2 - 4y_3 \geq -1 \\ & 0 \leq y_1, y_2, y_3 \end{array}$$

-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

$$\mathcal{P} \quad \begin{array}{ll} \text{maximize} & -4x_1 - 2x_2 - x_3 \\ \text{subject to} & -x_1 - x_2 + 2x_3 \leq -3 \\ & -4x_1 - 2x_2 + x_3 \leq -4 \\ & x_1 + x_2 - 4x_3 \leq 2 \\ & 0 \leq x_1, x_2, x_3 \end{array}$$

$$\mathcal{D} \quad \begin{array}{ll} \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ \text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\ & -y_1 - 2y_2 + y_3 \geq -2 \\ & 2y_1 + y_2 - 4y_3 \geq -1 \\ & 0 \leq y_1, y_2, y_3 \end{array}$$

-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

Not primal feasible.

Dual feasible!

The tableau below is said to be *dual feasible* because the objective row coefficients are all non-positive, but it is not *primal feasible*.

-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

The tableau below is said to be *dual feasible* because the objective row coefficients are all non-positive, but it is not *primal feasible*.

-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

A tableau is optimal if and only if it is both primal feasible **and** dual feasible.

The tableau below is said to be *dual feasible* because the objective row coefficients are all non-positive, but it is not *primal feasible*.

-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

A tableau is optimal if and only if it is both primal feasible **and** dual feasible.

Can we design a pivot for this tableau that tries to move it toward primal feasibility while retaining dual feasibility?

$\mathcal{D}$

$$
\begin{array}{lr}
\text{minimize} & -3y_1 - 4y_2 + 2y_3 \\
\text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\
 & -y_1 - 2y_2 + y_3 \geq -2 \\
 & 2y_1 + y_2 - 4y_3 \geq -1 \\
 & 0 \leq y_1, y_2, y_3
\end{array}
$$

$\mathcal{D}$

$$\begin{array}{lr} \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ \text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\ & -y_1 - 2y_2 + y_3 \geq -2 \\ & 2y_1 + y_2 - 4y_3 \geq -1 \\ & 0 \leq y_1, y_2, y_3 \end{array}$$

$$\left|\begin{array}{rrrrrr|r} -1 & -1 & 2 & 1 & 0 & 0 & -3 \\ -4 & -2 & 1 & 0 & 1 & 0 & -4 \\ 1 & 1 & -4 & 0 & 0 & 1 & 2 \\ \hline -4 & -2 & -1 & 0 & 0 & 0 & 0 \end{array}\right|$$

$\mathcal{D}$

$$
\begin{array}{lr}
\text{minimize} & -3y_1 - 4y_2 + 2y_3 \\
\text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\
 & -y_1 - 2y_2 + y_3 \geq -2 \\
 & 2y_1 + y_2 - 4y_3 \geq -1 \\
 & 0 \leq y_1, y_2, y_3
\end{array}
$$

$$
\left|\begin{array}{rrrrrr|r}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0
\end{array}\right|
$$

dual objective coefficients

$\mathcal{D}$

minimize $\quad -3y_1 - 4y_2 + 2y_3$

subject to

$$
\begin{aligned}
-y_1 - 4y_2 + y_3 &\geq -4 \\
-y_1 - 2y_2 + y_3 &\geq -2 \\
2y_1 + y_2 - 4y_3 &\geq -1 \\
0 &\leq y_1, y_2, y_3
\end{aligned}
$$

$$
\left|\begin{array}{cccccc|c}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0
\end{array}\right|
$$

dual objective coefficients

Dual variables

$\uparrow \quad \uparrow \quad \uparrow$

$y_1 \quad y_2 \quad y_3$

$\mathcal{D}$

$$\begin{array}{ll} \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ \text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\ & -y_1 - 2y_2 + y_3 \geq -2 \\ & 2y_1 + y_2 - 4y_3 \geq -1 \\ & 0 \leq y_1, y_2, y_3 \end{array}$$

$$\left|\begin{array}{cccccc|c} -1 & -1 & 2 & 1 & 0 & 0 & -3 \\ -4 & -2 & 1 & 0 & 1 & 0 & -4 \\ 1 & 1 & -4 & 0 & 0 & 1 & 2 \\ \hline -4 & -2 & -1 & 0 & 0 & 0 & 0 \end{array}\right|$$

dual objective coefficients

Dual variables

$$\uparrow \quad \uparrow \quad \uparrow$$
$$y_1 \quad y_2 \quad y_3$$

Increasing y_1 decreases the value of the dual objective.

Increasing y_1 means we pivot on row 1.

−1	−1	2	1	0	0	−3
−4	−2	1	0	1	0	−4
1	1	−4	0	0	1	2
−4	−2	−1	0	0	0	0

Increasing y_1 means we pivot on row 1.

$$
\begin{array}{|rrrrrr|r|l}
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}
$$

Increasing y_1 means we pivot on row 1.

-1	-1	2	1	0	0	-3	$\leftarrow$ pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

By how much can we increase the value of y_1?

Increasing y_1 means we pivot on row 1.

-1	-1	2	1	0	0	-3	$\leftarrow$ pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

By how much can we increase the value of y_1?

$$\begin{array}{r|}
-y_1 - 4y_2 + y_3 \geq -4 \\
-y_1 - 2y_2 + y_3 \geq -2 \\
2y_1 + y_2 - 4y_3 \geq -1
\end{array}$$

Increasing y_1 means we pivot on row 1.

$$\begin{array}{|rrrrrr|r|l} -1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\ -4 & -2 & 1 & 0 & 1 & 0 & -4 & \\ 1 & 1 & -4 & 0 & 0 & 1 & 2 & \\ \hline -4 & -2 & -1 & 0 & 0 & 0 & 0 & \end{array}$$

By how much can we increase the value of y_1?

$$\begin{array}{r|c} & \text{ratio} \\ -y_1 - 4y_2 + y_3 \geq -4 & 4/1 \\ -y_1 - 2y_2 + y_3 \geq -2 & 2/1 \\ 2y_1 + y_2 - 4y_3 \geq -1 & \end{array}$$

Increasing y_1 means we pivot on row 1.

$$\begin{array}{|rrrrrr|r|l} -1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\ -4 & -2 & 1 & 0 & 1 & 0 & -4 & \\ 1 & 1 & -4 & 0 & 0 & 1 & 2 & \\ \hline -4 & -2 & -1 & 0 & 0 & 0 & 0 & \\ \end{array}$$

$4/1 \quad 2/1$ ratios

By how much can we increase the value of y_1?

$$\begin{array}{r|c} & \text{ratio} \\ -y_1 - 4y_2 + y_3 \geq -4 & 4/1 \\ -y_1 - 2y_2 + y_3 \geq -2 & 2/1 \\ 2y_1 + y_2 - 4y_3 \geq -1 & \end{array}$$

Increasing y_1 means we pivot on row 1.

-1	$\boxed{-1}$	2	1	0	0	-3	$\leftarrow$ pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	
$4/1$	$2/1$						ratios

By how much can we increase the value of y_1?

	ratio
$-y_1 - 4y_2 + y_3 \geq -4$	$4/1$
$-y_1 - 2y_2 + y_3 \geq -2$	$2/1$
$2y_1 + y_2 - 4y_3 \geq -1$	

-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

-1	-1	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

-1	-1	2	1	0	0	-3	$\leftarrow$ pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

Any row having a negative rhs is a candidate pivot row.

-1	-1	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

Any row having a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

-1	-1	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

Any row having a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

The pivot column is given by the smallest ratio.

pivot
column
↓

-1	-1	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

Any row having a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

The pivot column is given by the smallest ratio.

pivot column ↓

-1	**[-1]**	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

Any row having a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

The pivot column is given by the smallest ratio.

-1	-1	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

1	1	-2	-1	0	0	3
-2	0	-3	-2	1	0	2
0	0	-2	1	0	1	-1
-2	0	-5	-2	0	0	6

-1	**[-1]**	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	-2	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

-1	**[-1]**	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-4	0	0	1	2	
-4	-2	-1	0	0	0	0	

1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	**[-2]**	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	**-2**	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	
1	1	0	-2	0	-1	4	
-2	0	0	-7/2	1	-3/2	7/2	
0	0	1	-1/2	0	-1/2	1/2	
-2	0	0	-9/2	0	-5/2	17/2	

1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	**[-2]**	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	
1	1	0	-2	0	-1	4	
-2	0	0	-7/2	1	-3/2	7/2	
0	0	1	-1/2	0	-1/2	1/2	
-2	0	0	-9/2	0	-5/2	17/2	optimal

1	1	0	-2	0	-1	4	
-2	0	0	-7/2	1	-3/2	7/2	
0	0	1	-1/2	0	-1/2	1/2	
-2	0	0	-9/2	0	-5/2	17/2	optimal

1	1	0	-2	0	-1	4	
-2	0	0	-7/2	1	-3/2	7/2	
0	0	1	-1/2	0	-1/2	1/2	
-2	0	0	-9/2	0	-5/2	17/2	optimal

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1/2 \end{pmatrix}$$

1	1	0	-2	0	-1	4	
-2	0	0	-7/2	1	-3/2	7/2	
0	0	1	-1/2	0	-1/2	1/2	
-2	0	0	-9/2	0	-5/2	17/2	optimal

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 0 \\ 5/2 \end{pmatrix},$$

1	1	0	-2	0	-1	4	
-2	0	0	-7/2	1	-3/2	7/2	
0	0	1	-1/2	0	-1/2	1/2	
-2	0	0	-9/2	0	-5/2	17/2	optimal

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 0 \\ 5/2 \end{pmatrix},$$

Optimal value $= -17/2$.

Apply the dual simplex algorithm to the following problem.

$$
\begin{array}{llrcl}
\mathcal{P} & \text{maximize} & -4x_1 - 2x_2 - x_3 & & \\
 & \text{subject to} & -x_1 - x_2 + 2x_3 & \leq & -3 \\
 & & -4x_1 - 2x_2 + x_3 & \leq & -4 \\
 & & x_1 + x_2 - x_3 & \leq & 2 \\
 & & 0 \leq x_1, x_2, x_3 & . &
\end{array}
$$

-1	[-1]	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	

-1	**[-1]**	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	
-2	0	-5	-2	0	0	6	

-1	**[-1]**	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

-1	**[-1]**	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

No negative entry in the pivot row!
What does this mean?

-1	[-1]	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

No negative entry in the pivot row!
What does this mean?

The dual problem is unbounded.

-1	[-1]	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

No negative entry in the pivot row!
What does this mean?

The dual problem is unbounded.
What can you say about the primal problem?

-1	**[-1]**	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

No negative entry in the pivot row!
What does this mean?

The dual problem is unbounded.
What can you say about the primal problem?
The primal is necessarily infeasible by the Weak Duality Theorem.

Solve the following LP using the dual simplex algorithm.

$$
\begin{array}{lrcrcrcr}
\text{maximize} & -4x_1 & - & 3x_2 & - & 2x_3 & & \\
\text{subject to} & x_1 & & & - & x_3 & \leq & -1 \\
 & -x_1 & - & x_2 & & & \leq & -2 \\
 & x_1 & - & x_2 & - & 2x_3 & \leq & 0 \\
 & 0 & \leq & x_1, & x_2, & x_3 & &
\end{array}
$$