Lecture 11: The Dual Simplex Algorithm

Math Dept, University of Washington
The Dual Simplex Algorithm
The Dual Simplex Algorithm

\[ \begin{aligned}
& \text{maximize} & -4x_1 - 2x_2 - x_3 \\
& \text{subject to} & -x_1 - x_2 + 2x_3 \leq -3 \\
& & -4x_1 - 2x_2 + x_3 \leq -4 \\
& & x_1 + x_2 - 4x_3 \leq 2 \\
& & 0 \leq x_1, x_2, x_3 \\
\end{aligned} \]

Not primal feasible.

Dual feasible!
The dual has feasible origin.
The Dual Simplex Algorithm

\[ \mathcal{P} \quad \text{maximize} \quad -4x_1 - 2x_2 - x_3 \]
subject to
\[ -x_1 - x_2 + 2x_3 \leq -3 \]
\[ -4x_1 - 2x_2 + x_3 \leq -4 \]
\[ x_1 + x_2 - 4x_3 \leq 2 \]
\[ 0 \leq x_1, x_2, x_3 \]

\[ \mathcal{D} \quad \text{minimize} \quad -3y_1 - 4y_2 + 2y_3 \]
subject to
\[ -y_1 - 4y_2 + y_3 \geq -4 \]
\[ -y_1 - 2y_2 + y_3 \geq -2 \]
\[ 2y_1 + y_2 - 4y_3 \geq -1 \]
\[ 0 \leq y_1, y_2, y_3 \]
The Dual Simplex Algorithm

\[ P \:\text{ maximize } \quad -4x_1 - 2x_2 - x_3 \]
subject to
\[ \begin{align*}
-x_1 - x_2 + 2x_3 &\leq -3 \\
-4x_1 - 2x_2 + x_3 &\leq -4 \\
x_1 + x_2 - 4x_3 &\leq 2 \\
0 &\leq x_1, x_2, x_3
\end{align*} \]

\[ D \:\text{ minimize } \quad -3y_1 - 4y_2 + 2y_3 \]
subject to
\[ \begin{align*}
-y_1 - 4y_2 + y_3 &\geq -4 \\
-y_1 - 2y_2 + y_3 &\geq -2 \\
2y_1 + y_2 - 4y_3 &\geq -1 \\
0 &\leq y_1, y_2, y_3
\end{align*} \]

\[
\begin{array}{ccccccc}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0
\end{array}
\]
The Dual Simplex Algorithm

\[ \mathcal{P} \text{ maximize } -4x_1 - 2x_2 - x_3 \]
\[ \text{subject to } -x_1 - x_2 + 2x_3 \leq -3 \]
\[ -4x_1 - 2x_2 + x_3 \leq -4 \]
\[ x_1 + x_2 - 4x_3 \leq 2 \]
\[ 0 \leq x_1, x_2, x_3 \]

\[ \mathcal{D} \text{ minimize } -3y_1 - 4y_2 + 2y_3 \]
\[ \text{subject to } -y_1 - 4y_2 + y_3 \geq -4 \]
\[ -y_1 - 2y_2 + y_3 \geq -2 \]
\[ 2y_1 + y_2 - 4y_3 \geq -1 \]
\[ 0 \leq y_1, y_2, y_3 \]

\[
\begin{array}{cccccc|c}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Not primal feasible.

Dual feasible!
The dual has feasible origin.
The Dual Simplex Algorithm

The tableau below is said to be dual feasible because the objective row coefficients are all non-positive, but it is not primal feasible.

\[
\begin{array}{cccccc|c}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

A tableau is optimal if and only if it is both primal feasible and dual feasible.
The Dual Simplex Algorithm

The tableau below is said to be *dual feasible* because the objective row coefficients are all non-positive, but it is not *primal feasible*.

\[
\begin{array}{cccccc|c}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

A tableau is optimal if and only if it is both primal feasible *and* dual feasible.
The Dual Simplex Algorithm

The tableau below is said to be dual feasible because the objective row coefficients are all non-positive, but it is not primal feasible.

| -1 | -1 | 2 | 1 | 0 | 0 | -3 |
| -4 | -2 | 1 | 0 | 1 | 0 | -4 |
| 1  | 1  | -4| 0 | 0 | 1 |  2 |
| -4 | -2 | -1| 0 | 0 | 0 |  0 |

A tableau is optimal if and only if it is both primal feasible and dual feasible.

Can we design a pivot for this tableau that tries to move it toward primal feasibility while retaining dual feasibility?
minimize \[-3y_1 - 4y_2 + 2y_3\]
subject to
\[-y_1 - 4y_2 + y_3 \geq -4\]
\[-y_1 - 2y_2 + y_3 \geq -2\]
\[2y_1 + y_2 + 4y_3 \geq -1\]
\[0 \leq y_1, y_2, y_3\]
minimize \(-3y_1 - 4y_2 + 2y_3\)

subject to

\(-y_1 - 4y_2 + y_3 \geq -4\)
\(-y_1 - 2y_2 + y_3 \geq -2\)
\(2y_1 + y_2 - 4y_3 \geq -1\)

\(0 \leq y_1, y_2, y_3\)
minimize $-3y_1 - 4y_2 + 2y_3$
subject to
$-y_1 - 4y_2 + y_3 \geq -4$
$-y_1 - 2y_2 + y_3 \geq -2$
$2y_1 + y_2 - 4y_3 \geq -1$
$0 \leq y_1, y_2, y_3$

$D$

dual objective coefficients

\[
\begin{array}{cccccc|c}
-3 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\end{array}
\]
\[\begin{align*}
\text{minimize} & \quad -3y_1 - 4y_2 + 2y_3 \\
\text{subject to} & \quad -y_1 - 4y_2 + y_3 \geq -4 \\
& \quad -y_1 - 2y_2 + y_3 \geq -2 \\
& \quad 2y_1 + y_2 - 4y_3 \geq -1 \\
& \quad 0 \leq y_1, y_2, y_3
\end{align*}\]

Dual variables

\[\begin{array}{cccccc|c}
& -1 & -1 & 2 & 1 & 0 & 0 \\
-3 & -4 & -2 & 1 & 0 & 1 & 0 \\
-4 & 1 & 1 & -4 & 0 & 0 & 1 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\end{array}\]

dual objective coefficients

\[\begin{array}{ccc}
\uparrow & \uparrow & \uparrow \\
y_1 & y_2 & y_3
\end{array}\]
The Dual Simplex Algorithm

\[ \begin{align*}
\text{minimize} & \quad -3y_1 - 4y_2 + 2y_3 \\
\text{subject to} & \quad -y_1 - 4y_2 + y_3 \geq -4 \\
& \quad -y_1 - 2y_2 + y_3 \geq -2 \\
& \quad 2y_1 + y_2 - 4y_3 \geq -1 \\
& \quad 0 \leq y_1, y_2, y_3
\end{align*} \]

Dual variables

Increasing \( y_1 \) decreases the value of the dual objective.
Increasing $y_1$ means we pivot on row 1.

\[
\begin{array}{cccccc}
-1 & -1 & 2 & 1 & 0 & 0 \\
-4 & -2 & 1 & 0 & 1 & 0 \\
1 & 1 & -4 & 0 & 0 & 1 \\
-4 & -2 & -1 & 0 & 0 & 0 \\
\end{array}
\begin{array}{c}
-3 \\
-4 \\
2 \\
0 \\
\end{array}
\]
Increasing $y_1$ means we pivot on row 1.

\[
\begin{array}{ccccc|c}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

← pivot row

By how much can we increase the value of $y_1$?
Increasing $y_1$ means we pivot on row 1.

\[
\begin{array}{cccccc}
-1 & -1 & 2 & 1 & 0 & 0 \\
-4 & -2 & 1 & 0 & 1 & 0 \\
1 & 1 & -4 & 0 & 0 & 1 \\
-4 & -2 & -1 & 0 & 0 & 0 \\
\end{array}
\]

$-3$ ← pivot row

$-4$ $-4$

$2$

$0$

By how much can we increase the value of $y_1$?
Increasing $y_1$ means we pivot on row 1.

\[
\begin{array}{rrrrrr|c}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

← pivot row

By how much can we increase the value of $y_1$?

\[
\begin{align*}
-y_1 - 4y_2 + y_3 & \geq -4 \\
-y_1 - 2y_2 + y_3 & \geq -2 \\
2y_1 + y_2 - 4y_3 & \geq -1
\end{align*}
\]
Increasing $y_1$ means we pivot on row 1.

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\begin{array}{cccccc|c}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[\text{← pivot row}\]

By how much can we increase the value of $y_1$?

\[
\begin{align*}
-y_1 - 4y_2 + y_3 & \geq -4 \\
y_1 - 2y_2 + y_3 & \geq -2 \\
2y_1 + y_2 - 4y_3 & \geq -1 \\
\end{align*}
\]

\[
\begin{array}{r|c}
\text{ratio} & \\
4/1 \\
2/1 \\
\end{array}
\]
The Dual Simplex Algorithm

Increasing $y_1$ means we pivot on row 1.

\[
\begin{array}{cccccc|c}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\hline
4/1 & 2/1 & \text{ratios} \\
\end{array}
\]

By how much can we increase the value of $y_1$?

\[
\begin{align*}
-y_1 - 4y_2 + y_3 & \geq -4 \\
-y_1 - 2y_2 + y_3 & \geq -2 \\
2y_1 + y_2 - 4y_3 & \geq -1 \\
\end{align*}
\]

\[
\text{ratio} \\
4/1 \\
2/1 \\
\]
Increasing $y_1$ means we pivot on row 1.

\[
\begin{array}{cccccc|c}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
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1 & 1 & -4 & 0 & 0 & 1 & 2 \\
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]

\[\text{← pivot row}\]

\[\text{ratios}\]

By how much can we increase the value of $y_1$?

\[
\begin{align*}
-y_1 - 4y_2 + y_3 & \geq -4 \\
-y_1 - 2y_2 + y_3 & \geq -2 \\
2y_1 + y_2 - 4y_3 & \geq -1
\end{align*}
\]

\[\begin{array}{c|c}
\text{ratio} & \\
4/1 & 2/1 \\
\end{array}\]
The Dual Simplex Algorithm

<table>
<thead>
<tr>
<th>-1  -1  2  1  0  0</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4 -2  1  0  1  0</td>
<td>-4</td>
</tr>
<tr>
<td>1  1  -4  0  0  1</td>
<td>2</td>
</tr>
<tr>
<td>----------------------</td>
<td>----</td>
</tr>
<tr>
<td>-4 -2  -1  0  0  0</td>
<td>0</td>
</tr>
</tbody>
</table>

Any row having a negative rhs is a candidate pivot row. Form the ratios with the negative entries in pivot row. The pivot column is given by the smallest ratio.
The Dual Simplex Algorithm

<table>
<thead>
<tr>
<th>-1</th>
<th>-1</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>-3</th>
<th>← pivot row</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>1</td>
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<td>-4</td>
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<tr>
<td>1</td>
<td>1</td>
<td>-4</td>
<td>0</td>
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<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
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<td></td>
</tr>
</tbody>
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The Dual Simplex Algorithm

<table>
<thead>
<tr>
<th>-1</th>
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<th>2</th>
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<th>0</th>
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<th>-3</th>
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</thead>
<tbody>
<tr>
<td>-4</td>
<td>-2</td>
<td>1</td>
<td>0</td>
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<td>2</td>
<td></td>
</tr>
<tr>
<td>-4</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Any row having a negative rhs is a candidate pivot row.
### The Dual Simplex Algorithm

Any row having a negative rhs is a candidate pivot row.

<table>
<thead>
<tr>
<th>-1</th>
<th>-1</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>-3 ← pivot row</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>1</td>
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<td>1</td>
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<td>-4</td>
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<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Form the ratios with the negative entries in pivot row.
The Dual Simplex Algorithm

<table>
<thead>
<tr>
<th>-1  -1  2  1  0  0</th>
<th>-3</th>
<th>← pivot row</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4  -2  1  0  1  0</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>1  1  -4  0  0  1  2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4  -2  -1  0  0  0  0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Any row having a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

The pivot column is given by the smallest ratio.
The Dual Simplex Algorithm

There are a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

The pivot column is given by the smallest ratio.
The Dual Simplex Algorithm

Any row having a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

The pivot column is given by the smallest ratio.
The Dual Simplex Algorithm

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</table>

| 1  | 1  | -2 | -1 | 0 | 0 | 3  |           |
| -2 | 0  | -3 | -2 | 1 | 0 | 2  |           |
| 0  | 0  | -2 | 1  | 0 | 1 | -1 |           |
| -2 | 0  | -5 | -2 | 0 | 0 | 6  |           |
The Dual Simplex Algorithm

<table>
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Lecture 11: The Dual Simplex Algorithm (Math Dept, University of Washington)
Math 407A: Linear Optimization
The Dual Simplex Algorithm

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</table>
|   |   |   |   |   |   |   |← pivot row

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## Lecture 11: The Dual Simplex Algorithm (Math Dept, University of Washington)

The Dual Simplex Algorithm is an algorithm used for solving linear programming problems. It is particularly useful when the initial feasible solution is not available or when the problem contains artificial variables.

### Algorithm Steps:
1. **Initialization**: Set up the initial tableau with the objective function and constraints.
2. **Pivot Selection**: Choose a pivot element to perform the pivot operation. This element is selected based on the rules of the dual simplex method.
3. **Pivot Operation**: Perform the pivot operation to update the tableau. This involves dividing the pivot row by the pivot element and then using it to eliminate other rows.
4. **Check for Optimal Solution**: Determine if the current tableau represents an optimal solution. This is done by checking if all the entries in the objective row are non-positive.

### Example:

Consider the following tableau:

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</table>

The tableau above represents a feasible solution. The pivot row is marked with an arrow pointing to the pivot element. The optimal solution is indicated by the last row, where all entries in the objective row are non-positive.
### The Dual Simplex Algorithm

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Optimal value = \(-\frac{17}{2}\).
### The Dual Simplex Algorithm

#### Table

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The optimal solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1/2 \end{pmatrix}$$

Optimal value = $-17/2$. 

---

Lecture 11: The Dual Simplex Algorithm (Math Dept, University of Washington)

Math 407A: Linear Optimization
The Dual Simplex Algorithm

\[
\begin{bmatrix}
1 & 1 & 0 & -2 & 0 & -1 \\
-2 & 0 & 0 & -7/2 & 1 & -3/2 \\
0 & 0 & 1 & -1/2 & 0 & -1/2 \\
-2 & 0 & 0 & -9/2 & 0 & -5/2 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
4 \\
1/2 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\end{bmatrix} =
\begin{bmatrix}
9/2 \\
0 \\
5/2 \\
\end{bmatrix},
\]

Optimal value = \(-17/2\).
The Dual Simplex Algorithm

\[
\begin{array}{cccc|c|c}
1 & 1 & 0 & -2 & 0 & -1 & 4 \\
-2 & 0 & 0 & -7/2 & 1 & -3/2 & 7/2 \\
0 & 0 & 1 & -1/2 & 0 & -1/2 & 1/2 \\
-2 & 0 & 0 & -9/2 & 0 & -5/2 & 17/2 \\
\end{array}
\]

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix} 0 \\ 4 \\ 1/2 \end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= \begin{pmatrix} 9/2 \\ 0 \\ 5/2 \end{pmatrix},
\]

Optimal value = \(-17/2\).
Apply the dual simplex algorithm to the following problem.

\[ \mathcal{P} \quad \text{maximize} \quad -4x_1 - 2x_2 - x_3 \]
\[ \text{subject to} \quad -x_1 - x_2 + 2x_3 \leq -3 \]
\[ -4x_1 - 2x_2 + x_3 \leq -4 \]
\[ x_1 + x_2 - x_3 \leq 2 \]
\[ 0 \leq x_1, x_2, x_3 \].
The Dual Simplex Algorithm

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No negative entry in the pivot row!

What does this mean?

The dual problem is unbounded.

What can you say about the primal problem?

The primal is necessarily infeasible by the Weak Duality Theorem.
The Dual Simplex Algorithm

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The Dual Simplex Algorithm

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No negative entry in the pivot row!

What does this mean?
The dual problem is unbounded.

What can you say about the primal problem?
The primal is necessarily infeasible by the Weak Duality Theorem.
### The Dual Simplex Algorithm

The dual problem is unbounded.

What can you say about the primal problem?

The primal is necessarily infeasible by the Weak Duality Theorem.

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What does this mean?

The dual problem is unbounded.
The Dual Simplex Algorithm

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What does this mean?

The dual problem is unbounded.
What can you say about the primal problem?
The Dual Simplex Algorithm

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What does this mean?

The dual problem is unbounded.
What can you say about the primal problem?
The primal is necessarily infeasible by the Weak Duality Theorem.
Solve the following LP using the dual simplex algorithm.

maximize \(-4x_1 - 3x_2 - 2x_3\)

subject to

\[
\begin{align*}
    x_1 & - x_3 & \leq & -1 \\
    -x_1 & - x_2 & \leq & -2 \\
    x_1 & - x_2 & - 2x_3 & \leq 0 \\
    0 & \leq & x_1, x_2, x_3 &
\end{align*}
\]