

Math 407A: Linear Optimization

Lecture 10: General Duality Theory

Math Dept, University of Washington

- 1 General Duality Theory
- 2 General Weak Duality theorem
- 3 Theorems of the Alternative

General Duality Theory

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The great advantage of doing this is that it allows the modeler to understand the nature of the dual variables in terms of the original problem statement and the original decision variables.

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In our discussion we still need to make use of a *standard form* but it will be much more general and flexible than the standard form used so far.



Expanded Standard Form for General Duality Theory

$$\begin{aligned} \mathcal{P} \quad & \text{maximize} && \sum_{j=1}^n c_j x_j \\ & \text{subject to} && \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i \in I \\ & && \sum_{j=1}^n a_{ij} x_j = b_i \quad i \in E \\ & && 0 \leq x_j \quad j \in R . \end{aligned}$$

Here the index sets I , E , and R are such that

$$I \cap E = \emptyset, \quad I \cup E = \{1, 2, \dots, m\}, \quad \text{and} \quad R \subset \{1, 2, \dots, n\}.$$



Primal-Dual Correspondences

In the Dual	In the Primal
	Maximization

Primal-Dual Correspondences

In the Dual	In the Primal
Minimization	Maximization

Primal-Dual Correspondences

In the Dual	In the Primal
Minimization	Maximization Inequality Constraints

Primal-Dual Correspondences

In the Dual	In the Primal
Minimization	Maximization
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$F = \{1, 2, \dots, n\} \setminus R =$ the free variables.

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Example: General Duality

Compute the dual of the LP

$$\begin{array}{ll} \text{maximize} & x_1 - 2x_2 + 3x_3 \\ \text{subject to} & 5x_1 + x_2 - 2x_3 \leq 8 \\ & -x_1 + 5x_2 + 8x_3 = 10 \\ & x_1 \leq 10, 0 \leq x_3 \end{array}$$

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$$\text{minimize} \quad 8y_1 + 10y_2 + 10y_3$$

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$$\begin{array}{ll} \text{minimize} & 8y_1 + 10y_2 + 10y_3 \\ \text{subject to} & 5y_1 - y_2 + y_3 = 1 \end{array}$$

$$0 \leq y_1, 0 \leq y_3$$



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$$\begin{array}{llll} \text{minimize} & 8y_1 + 10y_2 + 10y_3 & & \\ \text{subject to} & 5y_1 - y_2 + y_3 & = 1 & \\ & y_1 + 5y_2 & = -2 & \\ & 0 \leq y_1, 0 \leq y_3 & & \end{array}$$



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$$\begin{array}{llll} \text{minimize} & 8y_1 + 10y_2 + 10y_3 & & \\ \text{subject to} & 5y_1 - y_2 + y_3 & = 1 & \\ & y_1 + 5y_2 & = -2 & \\ & -2y_1 + 8y_2 & \geq 3 & \\ & 0 \leq y_1, 0 \leq y_3 & & \end{array}$$



Second Example: General Duality

$$\text{maximize} \quad 2x_1 - 3x_2 + x_3$$

$$\text{subject to} \quad x_1 + 5x_2 - 2x_3 = 4$$

$$10x_1 + x_2 - 5x_3 \leq 20$$

$$5x_1 - x_2 - x_3 = 3$$

$$x_1 \leq 6, \quad 0 \leq x_2$$



Second Example: Solution

Primal

$$\begin{array}{rllll} \text{maximize} & 2x_1 & -3x_2 & +x_3 & & \\ \text{subject to} & x_1 & +5x_2 & -2x_3 & = & 4 \\ & 10x_1 & +x_2 & -5x_3 & \leq & 20 \\ & 5x_1 & -x_2 & -x_3 & = & 3 \\ & x_1 & \leq 6, & 0 & \leq & x_2 \end{array}$$

Dual

$$\begin{array}{rllll} \text{minimize} & 4y_1 & +20y_2 & +3y_3 & +6y_4 & \\ \text{subject to} & y_1 & +10y_2 & +5y_3 & +y_4 & = 2 \\ & 5y_1 & +y_2 & -y_3 & & \geq -3 \\ & -2y_1 & -5y_2 & -y_3 & & = 1 \\ & 0 & \leq y_2, & 0 & \leq & y_4 \end{array}$$

General Weak Duality theorem

Theorem: Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ is feasible for \mathcal{P} and $y \in \mathbb{R}^m$ is feasible for \mathcal{D} , then

$$c^T x \leq y^T A x \leq b^T y.$$

Moreover, the following statements hold.



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Moreover, the following statements hold.

(i) If \mathcal{P} is unbounded, then \mathcal{D} is infeasible.

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Moreover, the following statements hold.

- (i) If \mathcal{P} is unbounded, then \mathcal{D} is infeasible.
- (ii) If \mathcal{D} is unbounded, then \mathcal{P} is infeasible.
- (iii) If \bar{x} is feasible for \mathcal{P} and \bar{y} is feasible for \mathcal{D} with $c^T \bar{x} = b^T \bar{y}$, then \bar{x} is an optimal solution to \mathcal{P} and \bar{y} is an optimal solution to \mathcal{D} .

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Proof: $x \in \mathbb{R}^n$ is feasible for \mathcal{P} and $y \in \mathbb{R}^m$ is feasible for \mathcal{D} .

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$$c^T x = \sum_{j \in R} c_j x_j + \sum_{j \in F} c_j x_j$$

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(Since $c_j \leq \sum_{i=1}^m a_{ij} y_i$ and $x_j \geq 0$ for $j \in R$)

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General Weak Duality theorem

$$x^T Ay$$

General Weak Duality theorem

$$x^T Ay = \sum_{i \in I} \left(\sum_{j=1}^n a_{ij} x_j \right) y_i + \sum_{i \in E} \left(\sum_{j=1}^n a_{ij} x_j \right) y_i$$

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(Since $\sum_{j=1}^n a_{ij} x_j \leq b_i$ and $0 \leq y_i$ for $i \in I$)

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$$= \sum_{i=1}^m b_i y_i$$

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(Since $\sum_{j=1}^n a_{ij} x_j \leq b_i$ and $0 \leq y_i$ for $i \in I$
and $\sum_{j=1}^n a_{ij} x_j = b_i$ for $i \in E$.)

$$\begin{aligned}&= \sum_{i=1}^m b_i y_i \\ &= b^T y.\end{aligned}$$



Systems of Equations and Inequalities

Let $g \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$.

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Let $g \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$.

Question: Does there exist $x \in \mathbb{R}^n$ such that

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What does this say about the dual to this LP?



Systems of Equations and Inequalities

The dual to the LP

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What is the relationship between these two LPs?



A Theorem of the Alternative

Theorem: *Either there exists a solution $x \in \mathbb{R}^n$ to the system*

$$0 \leq x, \quad g^T x < 0, \quad \text{and} \quad Ax = 0$$

or there exists a solution $y \in \mathbb{R}^m$ to the system

$$0 \leq g + A^T y,$$

but not both.