

# Linear Programming

## Lecture 1: Linear Algebra Review

- 1 Linear Algebra Review
- 2 Linear Algebra Review
- 3 Block Structured Matrices
- 4 Gaussian Elimination Matrices
- 5 Gauss-Jordan Elimination (Pivoting)

# Matrices in $\mathbb{R}^{m \times n}$

$$A \in \mathbb{R}^{m \times n}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

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# Matrix Vector Multiplication

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

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$$= x_1 \mathbf{a}_{\bullet 1} + x_2 \mathbf{a}_{\bullet 2} + \cdots + x_n \mathbf{a}_{\bullet n}$$

A linear combination of the columns.

# The Range of a Matrix

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# Two Special Subspaces

Let  $v_1, \dots, v_k \in \mathbb{R}^n$ .

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- The linear span of  $v_1, \dots, v_k$ :

$$\text{Span}[v_1, \dots, v_k] = \{y \mid y = \xi_1 v_1 + \xi_2 v_2 + \dots + \xi_k v_k, \xi_1, \dots, \xi_k \in \mathbb{R}\}$$

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- The subspace orthogonal to  $v_1, \dots, v_k$ :

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# Matrix Vector Multiplication

A row space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1\bullet} \bullet x \\ a_{2\bullet} \bullet x \\ \vdots \\ a_{m\bullet} \bullet x \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i}x_i \\ \sum_{i=1}^n a_{2i}x_i \\ \vdots \\ \sum_{i=1}^n a_{mi}x_i \end{bmatrix}$$

The dot product of  $x$  with the rows of  $A$ .



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**Fundamental Theorem of the Alternative:**

$$\text{Nul}(A) = \text{Ran}(A^T)^\perp \quad \text{Ran}(A) = \text{Nul}(A^T)^\perp$$

# Block Structured Matrices

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

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where

$$B = \begin{bmatrix} 3 & -4 & 1 \\ 2 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

# Multiplication of Block Structured Matrices

Consider the matrix product  $AM$ , where

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# Multiplication of Block Structured Matrices

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$$AM = \begin{bmatrix} B & I_{3 \times 3} \\ 0_{2 \times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix}$$

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# Solving Systems of Linear equations

Let  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ .

Find all solutions  $x \in \mathbb{R}^n$  to the system  $Ax = b$ .

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If a solution  $x_0 \in \mathbb{R}^n$  exists, then the set of solutions is given by

$$x_0 + \text{Nul}(A) .$$

# Gaussian Elimination and the 3 Elementary Row Operations

We solve the system  $Ax = b$  by transforming the augmented matrix

$$[A | b]$$

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These elementary row operations can be interpreted as multiplying the augmented matrix on the left by a special **nonsingular** matrix.

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$$AN = \begin{bmatrix} a_{1\bullet}N \\ a_{2\bullet}N \\ \vdots \\ a_{m\bullet}N \end{bmatrix}$$



# Gaussian Elimination Matrices

The key step in Gaussian elimination is to transform a vector of the form

$$\begin{bmatrix} a \\ \alpha \\ b \end{bmatrix},$$

where  $a \in \mathbb{R}^k$ ,  $0 \neq \alpha \in \mathbb{R}$ , and  $b \in \mathbb{R}^{n-k-1}$ , into one of the form

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This can be accomplished by left matrix multiplication as follows.

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Note that a Gaussian elimination matrix and its inverse are both lower triangular matrices.

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Transformation to echelon (upper triangular) form.



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# Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

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$$\begin{bmatrix} I_{k \times k} & -\alpha^{-1} \mathbf{a} & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1} \mathbf{b} & I_{(n-k-1) \times (n-k-1)} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \alpha \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

What is the inverse of this matrix?

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$$\begin{bmatrix} I_{k \times k} & \mathbf{a} & 0 \\ 0 & \alpha & 0 \\ 0 & \mathbf{b} & I_{(n-k-1) \times (n-k-1)} \end{bmatrix}.$$