Linear Programming [1]

Lecture 1: Linear Algebra Review

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Matrices in $\mathbb{R}^{m \times n}$

$A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

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columns

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Matrix Vector Multiplication

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

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$$= x_1\, a_{\bullet 1} \; + \; x_2\, a_{\bullet 2} \; + \; \cdots \; + \; x_n\, a_{\bullet n}$$

A linear combination of the columns.

The Range of a Matrix

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Two Special Subspaces

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- The subspace orthogonal to $v_1, \ldots, v_k$:

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The dot product of x with the rows of A.

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$$\begin{aligned} \mathrm{Nul}\,(A) &= \text{subspace orthogonal to the rows of } A \\ &= \mathrm{Span}\,[a_{1\bullet}, a_{2\bullet}, \ldots, a_{m\bullet}]^{\perp} \end{aligned}$$

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Fundamental Theorem of the Alternative:

$$\mathrm{Nul}\,(A) = \mathrm{Ran}\left(A^T\right)^{\perp} \qquad \mathrm{Ran}\,(A) = \mathrm{Nul}\left(A^T\right)^{\perp}$$

Block Structured Matrices

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

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where

$$B = \begin{bmatrix} 3 & -4 & 1 \\ 2 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

Multiplication of Block Structured Matrices

Consider the matrix product AM, where

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}$$

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Can we exploit the structure of A?

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$$= \begin{bmatrix} \begin{bmatrix} 2 & -11 \\ 2 & 12 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix} \end{bmatrix}$$

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$$
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&= \begin{bmatrix} 4 & -12 \\ 6 & 15 \\ 1 & -2 \\ 4 & 1 \\ -4 & -1 \end{bmatrix}.
\end{aligned}
$$

Solving Systems of Linear equations

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Find all solutions $x \in \mathbb{R}^{n}$ to the system $Ax = b$.

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The solution set is either empty, a single point, or an infinite set.

If a solution $x_0 \in \mathbb{R}^n$ exists, then the set of solutions is given by

$$x_0 \; + \; \mathrm{Nul}\,(A) \; .$$

Gaussian Elimination and the 3 Elementary Row Operations

We solve the system $Ax = b$ by transforming the augmented matrix

$$[\, A \,|\, b]$$

into upper echelon form using the three elementary row operations.

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We solve the system $Ax = b$ by transforming the augmented matrix

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These elementary row operations can be interpreted as multiplying the augmented matrix on the left by a special matrix.

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Multiplying any $4 \times n$ matrix on the left by the exchange matrix

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Notes on Matrix Multiplication

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$$AN = \begin{bmatrix} a_{1\bullet}N \\ a_{2\bullet}N \\ \vdots \\ a_{m\bullet}N \end{bmatrix}$$

Gaussian Elimination Matrices

The key step in Gaussian elimination is to transform a vector of the form

$$\begin{bmatrix} a \\ \alpha \\ b \end{bmatrix},$$

where $a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$, into one of the form

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This can be accomplished by left matrix multiplication as follows.

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$a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$

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Note that a Gaussian elimination matrix and its inverse are both lower triangular matrices.

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Transformation to echelon (upper triangular) form.

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Gauss-Jordan Elimination, or Pivot Matrices

What happens in the following multiplication?

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