The results contained in these notes refer only to LPs having the following standard form:

\[ \begin{align*}
\mathcal{P} \quad & \text{maximize} & c^T x \\
\text{subject to} & & Ax \leq b, \ 0 \leq x
\end{align*} \]

and its dual

\[ \begin{align*}
\mathcal{D} \quad & \text{minimize} & b^T y \\
\text{subject to} & & A^T y \geq c, \ 0 \leq y
\end{align*} \]

where \( A \in \mathbb{R}^{m \times n} \), \( c \in \mathbb{R}^n \), and \( b \in \mathbb{R}^n \).

**Theorem** [The Weak Duality Theorem of Linear Programming]
If \( x \) is feasible for \( \mathcal{P} \) and \( y \) is feasible for \( \mathcal{D} \), then

\[ c^T x \leq y^T Ax \leq y^T b, \]

with \( c^T x = y^T b \) only if \( x \) solve \( \mathcal{P} \) and \( y \) solves \( \mathcal{D} \). In particular, if \( \mathcal{P} \) is unbounded, then \( \mathcal{D} \) is necessarily infeasible, and if \( \mathcal{D} \) is unbounded, then \( \mathcal{P} \) is necessarily unfeasible.

**Proof:** Let \( x \in \mathbb{R}^n \) be feasible for \( \mathcal{P} \) and \( y \in \mathbb{R}^m \) be feasible for \( \mathcal{D} \). Then, for each \( j = 1, 2, \ldots, n \), we have

\[ c_j x_j \leq \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j, \quad \text{since} \quad 0 \leq x_j \quad \text{and} \quad c_j \leq \sum_{i=1}^{m} a_{ij} y_i. \]

Summing the terms \( c_j x_j \) over \( j \) gives the inequality

\[ \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j. \quad (1) \]

Similarly, for each \( i = 1, 2, \ldots, m \), we have

\[ \left( \sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq b_i y_i, \quad \text{since} \quad 0 \leq y_i \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i. \]

Summing the terms \( b_i y_i \) over \( i \) gives the inequality

\[ \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{m} b_i y_i. \quad (2) \]

Using (1) and (2) along with the relation

\[ \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j = y^T Ax = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} x_j \right) y_i, \]

we obtain

\[ c^T x = \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j \quad \text{[by (1) above]} \]

\[ = y^T Ax \]

\[ = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{m} b_i y_i \quad \text{[by (2) above]} \]

\[ = b^T y. \]

Now if \( c^T x = b^T y \), then for any other primal feasible point \( \hat{x} \) and any other dual feasible point \( \hat{y} \), we have

\[ c^T \hat{x} \leq b^T \hat{y} = c^T x \leq b^T y. \]
Therefore, \( c^T \hat{x} \leq c^T x \) for all primal feasible \( \hat{x} \), i.e. \( x \) solves \( P \), and \( b^T y \leq b^T \hat{y} \) for all dual feasible \( \hat{y} \), i.e. \( y \) solves \( D \).

Two Phase Simplex Algorithm

Let

\[
T_0 = \begin{bmatrix}
A & I & b \\
c^T & 0 & 0
\end{bmatrix}
\]

be the initial simplex tableau for \( P \). Recall that if this tableau is primal feasible (\( b \geq 0 \)), then it defines an initial basic feasible solution and we can proceed with the simplex algorithm to compute an optimal solution to \( P \) or determine that \( P \) is unbounded. Similarly, if the initial simplex tableau \( T_0 \) is dual feasible (\( c \leq 0 \)), then it defines an initial basic feasible solution for the dual \( D \) and we can proceed with the dual simplex algorithm to compute an optimal solution to \( P \) or determine that \( P \) is infeasible (\( D \) is unbounded). However, if the initial tableau \( T_0 \) is neither primal nor dual feasible, then we must use the two phase simplex algorithm to solve it.

The two phase simplex algorithm starts by first solving the auxiliary LP

\[
\begin{align*}
\mathcal{A} \quad & \text{minimize} & x_0 \\
\text{subject to} & & Ax - x_0 e \leq b, \\ & & 0 \leq x_0, \\ & & 0 \leq x,
\end{align*}
\]

where the vector \( e \in \mathbb{R}^m \) is the vector of all ones. If the optimal value in \( \mathcal{A} \) is zero, then the optimal solution defines an initial feasible tableau for \( P \) to which the primal simplex algorithm can be applied. If the optimal value in \( \mathcal{A} \) is positive, then \( P \) is infeasible. The following is a concise description of the two phase simplex algorithm implemented with an anti-cycling rule.

The Two Phase Simplex Algorithm Outcomes

**Phase I:** Formulate and solve the auxiliary problem. Two outcomes are possible:

(i) The optimal value in the auxiliary problem is positive. In this case the original problem is infeasible.

(ii) The optimal value is feasible and an initial feasible tableau for the original problem is obtained.

**Phase II:** If the original problem is feasible, apply the simplex algorithm to the initial feasible tableau obtained from Phase I above. Again, two outcomes are possible:

(i) The LP is determined to be unbounded.

(ii) An optimal basic feasible solution is obtained.

Therefore, we see that the two phase simplex algorithms can be applied to solve any LP. This yields the following theorem.

**Theorem** [The Fundamental Theorem of Linear Programming] Every LP has the following three properties:

(i) If it has no optimal solution, then it is either infeasible or unbounded.

(ii) If it has a feasible solution, then it has a basic feasible solution.

(iii) If it is bounded, then it has an optimal basic feasible solution.

**Proof:** The two phase simplex algorithm is always finitely terminating. By applying it to any instance of \( P \), we verify the statements of this theorem.

The two phase simplex algorithm can also be used as a basis for the proof of the strong duality theorem of linear programming.

**Theorem** [The Strong Duality Theorem] If either \( P \) or \( D \) has a finite optimal value, then so does the other and these optimal values coincide in which case optimal solutions to both \( P \) and \( D \) exist.

**Proof:** Since the dual of the dual is the primal, we may assume with no loss in generality that the primal problem has a finite optimal value. Then, by the Fundamental Theorem of Linear Programming, an optimal basic feasible solution \( \hat{x} \) must exist. Let \( B \) be a basis associated with \( \hat{x} \) whose corresponding simplex tableau
identifies $\hat{x}$ as optimal, that is, the simplex tableau corresponding to this basis is both primal and dual feasible. Denote this tableau by

$$
\begin{bmatrix}
RA & Rb \\
(c - A^Ty) & -y^T & -y^Tb
\end{bmatrix}.
$$

Since this simplex tableau is both primal and dual feasible, we have, in particular, that the entries $c - A^Ty$ and $-y$ in the cost row of this tableau must all be non-positive, i.e.

$$0 \leq y \quad \text{and} \quad A^Ty \geq c.$$

Hence $y$ is feasible for $D$ with

$$y^Tb = c^T\hat{x} = \text{optimal value in } P.$$

Therefore, by the Weak Duality Theorem for linear programming, $y$ must solve $D$. □

The strong duality theorem implies that boundedness of either $P$ or $D$ implies that optimal solutions to both exist and the optimal values coincide. This in turn implies equality in the weak duality theorem. Hence, the boundedness of either $P$ or $D$ occurs if and only if there exist a primal-dual feasible pair $(x, y)$ for which $c^Tx = y^TAx = y^Tb$. This observation gives us the following theorem.

**Theorem** [The Complementary Slackness Theorem] The vector $x \in \mathbb{R}^n$ solves $P$ and the vector $y \in \mathbb{R}^m$ solves $D$ if and only if $x$ is feasible for $P$ and $y$ is feasible for $D$ and

1. either $0 = x_j$ or $\sum_{i=1}^{m} a_{ij}y_i = c_j$ or both for $j = 1, \ldots, n$, and
2. either $0 = y_i$ or $\sum_{j=1}^{n} a_{ij}x_j = b_i$ or both for $i = 1, \ldots, m$.

**Proof:** If $x$ solves $P$ and $y$ solves $D$, then by the Strong Duality Theorem we have equality in the Weak Duality Theorem: $c^Tx = y^TAx = y^Tb$. We now examine the consequences of these two equations. First note that the equation $c^Tx = y^TAx$ implies that

$$0 = x^T(A^Ty - c) = \sum_{j=1}^{n} x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j).$$

Since feasibility implies that

$$0 \leq x_j \quad \text{and} \quad 0 \leq \sum_{i=1}^{m} a_{ij}y_i - c_j$$

for $j = 1, \ldots, n$, we must have that

$$x_j(\sum_{i=1}^{m} a_{ij}y_i - c_j) = 0 \quad \text{for} \quad j = 1, \ldots, n,$$

or equivalently,

$$x_j = 0 \quad \text{or} \quad \sum_{i=1}^{m} a_{ij}y_i = c_j \quad \text{or both}$$

for $j = 1, \ldots, n$. This is statement (i) of the Theorem.

Similarly, the equation $y^TAx = y^Tb$ implies that

$$0 = y^T(b - Ax) = \sum_{i=1}^{m} y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j).$$

Again, feasibility implies that

$$0 \leq y_i \quad \text{and} \quad 0 \leq b_i - \sum_{j=1}^{n} a_{ij}x_j$$

for $i = 1, \ldots, m$. Thus, we must have

$$y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j) = 0 \quad \text{for} \quad i = 1, \ldots, m.$$
or equivalently,

\[ y_i = 0 \quad \text{or} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{or both} \]

for \( i = 1, \ldots, m \). This is statement (ii) of the Theorem.

Conversely, if (i) and (ii) are satisfied, then we get equality in the Weak Duality Theorem, which implies optimality. \( \square \)

### Sensitivity Analysis

Let \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), and \( c \in \mathbb{R}^n \). This data defines an LP in standard form by

\[
P \quad \text{maximize} \quad c^T x \\
\text{subject to} \quad Ax \leq b, \ 0 \leq x.
\]

We associate \( P \) the optimal value function \( V : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\pm \infty\} \) defined by

\[
V(u) = \max \{c^T x : Ax \leq b + u \} \quad \text{subject to} \quad 0 \leq x
\]

for all \( u \in \mathbb{R}^m \). Let

\[ \mathcal{F}(u) = \{ x \in \mathbb{R}^n : Ax \leq b + u, \ 0 \leq x \} \]

denote the feasible region for the LP associated with value \( V(u) \). If \( \mathcal{F}(u) = \emptyset \) for some \( u \in \mathbb{R}^m \), we define \( V(u) = -\infty \).

**Theorem** (Fundamental Theorem on Sensitivity Analysis)

If \( P \) is primal nondegenerate, i.e. the optimal value is finite and no basic variable in any optimal tableau takes the value zero, then the dual solution \( y^* \) is unique and there is an \( \epsilon > 0 \) such that

\[
V(u) = b^T y^* + u^T y^* \quad \text{whenever} \quad |u_i| \leq \epsilon, \ i = 1, \ldots, m.
\]

Thus, in particular, the optimal value function \( V \) is differentiable at \( u = 0 \) with \( \nabla V(0) = y^* \).

**Proof:** Let

\[
\begin{bmatrix}
RA & R & Rb \\
(c - A^T y^*)^T & -(y^*)^T & -b^T y^*
\end{bmatrix}
\]

be any optimal tableau for \( P \). Primal nondegeneracy implies that every component of the vector \( Rb \) is strictly positive. If there is another dual optimal solution \( \tilde{y} \) associated with an tableau, then we can pivot to it using simplex pivots. All of these simplex pivots must be degenerate since the optimal value cannot change. But degenerate pivots can only be performed if the tableau is degenerate, i.e. there is an index \( i \) such that \((Rb)_i = 0\). But then the basic variable associated with \((Rb)_i\) must take the value zero contradicting the hypothesis that \( Rb \) is a strictly positive vector. Hence the only possible optimal tableau is the one given.

The only other way to have multiple dual solutions is if there is an unbounded ray of optimal solutions emanating from the optimal solution identified by the unique optimal tableau. For this to occur, there must be a row in the optimal tableau such that any positive multiple of that row can be added to the cost row without changing the optimal value. Again, this can only occur if some \((Rb)_i\) is zero leading to the same contradiction. Therefore, primal nondegeneracy implies the uniqueness of the dual solution \( y^* \).

Next let \( 0 < \delta < \min \{(Rb)_i | i = 1, \ldots, m\} \). Due to the continuity of the mapping \( u \rightarrow Ru \), there is an \( \epsilon > 0 \) such that \(|(Ru)_i| \leq \delta \ i = 1, \ldots, m \) whenever \(|u_j| \leq \epsilon \ j = 1, \ldots, n \). Hence, if we perturb \( b \) by \( u \), then

\[
R(b + u) = Rb + Ru \geq Rb - \epsilon e > 0
\]

whenever \(|u_j| \leq \epsilon \ j = 1, \ldots, n \). Therefore, if we perturb \( b \) by \( u \) in the optimal tableau with \(|u_j| \leq \epsilon \ j = 1, \ldots, n \), we get the tableau

\[
\begin{bmatrix}
RA & R & Rb + Ru \\
(c - A^T y^*)^T & -(y^*)^T & -b^T y^* - b^T u
\end{bmatrix}
\]

which is still both primal and dual feasible, hence optimal with optimal value \( V(u) = b^T y^* + b^T u \) proving the theorem. \( \square \)