Computing Solutions sets to $Ax = b$ when $m < n$

A number of students have asked me to describe the procedure for representing the set of solutions to the linear system $Ax = b$ when $m < n$. I will give a example illustrating this procedure below, but I begin with some general observations. First, since $m < n$ we know that the null space of A is nontrivial, that is, it has dimension greater than or equal to $n - m \geq 1$. Hence, if one solution to $Ax = b$ say x_0, then there must be infinitely many solutions since the set of solution is given by $\{x_0 + z \mid Az = 0\} = x_0 + \mathrm{Nul(A)}$. This motivates the representation for the solution set that I am looking for. First, find one solution x_0 to $Ax = b$, then compute a basis for Nul(A), say $v_1, v_2, \ldots, v_k$. Then $\{x \mid Ax = b\} = x_0 + \mathrm{Span[v_1, v_2, \ldots, v_k] = x_0 + Ran(V)}$, where $V = [v_1, v_2, \ldots, v_k]$ the matrix whose columns are given by the vectors $v_1, v_2, \ldots, v_k$. These two step can be performed simultaneously by computing the reduced echelon form for the associated augmented matrix.

Example: Give a representation for the set of solutions to the linear system

$$\begin{bmatrix} 1 & 1 & 1 & 4 & 5 \\ 3 & 2 & -1 & 8 & 5 \\ 1 & 2 & -1 & 2 & 3 \end{bmatrix} x = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$

Solution: The associated augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 4 & 5 & 5 \\ 3 & 2 & -1 & 8 & 5 & 1 \\ 1 & 2 & -1 & 2 & 3 & 1 \end{array}\right].$$

A reduced echelon form for this augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right].$$

Hence $x_0 = [0, 2, 3, 0, 0]^T$ solves the system, and the nullity of the matrix A is 2. Now think of the reduced echelon matrix above as having block structure

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right] = [I\ T|x_0]$$

and observe that the columns of the block matrix $\begin{bmatrix} T \\ -I \end{bmatrix}$ are necessarily in the null space of the block matrix $[I\ T]$ since $[I\ T]\begin{bmatrix} T \\ -I \end{bmatrix} = 0$. But $\begin{bmatrix} T \\ -I \end{bmatrix}$ has two linearly independent columns, and so they must form a basis for the null space of $[I\ T]$, or equivalently A. Hence the set of solutions to this system is given by

$$\begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \mathrm{Span}\left[\begin{pmatrix} 3 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \\ -1 \end{pmatrix}\right].$$

Of course, you should double check all of this by plugging x_0 to see that the equation is satisfied as well as plug in the two vectors spanning the null space to see that you get the zero vector.

Two further problems appear on the next page for you to try.

(1) Show that the set of solutions to the linear system

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

is given by

$$\begin{pmatrix} 5 \\ -4 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \mathrm{Span}\left[\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}\right].$$

(2) (This problem is trickier since you have to think about rearranging columns) Show that the set of solutions to the linear system

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

is given by

$$\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mathrm{Span}\left[\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}\right].$$