Math 407 Section A

SAMPLE PROBLEMS FOR THE FIRST QUIZ

1. Consider the system

$$\begin{array}{rcrcrcr} 4x_1 & & & - & x_3 & = & 200 \\ 9x_1 & + & x_2 & - & x_3 & = & 200 \\ 7x_1 & - & x_2 & + & 2x_3 & = & 200 \,. \end{array}$$

   Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 30 \\ -150 \\ -80 \end{pmatrix}$$

2. Represent the linear span of the four vectors

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 1 \\ 7 \\ 1 \end{bmatrix}, \quad \text{and} \quad x_4 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 5 \end{bmatrix},$$

   as the range space of some matrix.

   Solution The span is the range of A where

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 2 & 1 & 7 & 0 \\ 1 & -2 & 1 & 5 \end{bmatrix}$$

   A null space basis is given by the columns of the matrix

$$\begin{bmatrix} 3 & 1 \\ 1 & -2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

3. Compute a basis for $\text{Nul}\,(A^T)^\perp$ where A is given by

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 2 & 1 & 7 & 0 \\ 1 & -2 & 1 & 5 \end{bmatrix}.$$

   Solution Since $\text{Nul}\,(A^T)^\perp = \text{Ran}(A)$ and the range of A is the column space of A, we need only select linearly independent columns from A. Alternatively, we can row reduce A^T. If we do this, we get only 2 linearly independent vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

   which form the desired basis.

4. Find the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -4 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

Solution $\quad B^{-1} = \frac{1}{4}\begin{pmatrix} 6 & 2 & -2 \\ -1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$.

5. Compute a basis for the null space of the matrix

$$\begin{bmatrix} 2 & 1 & 3 & 4 & 5 \\ 1 & 3 & 2 & 7 & 8 \end{bmatrix}.$$

Solution First compute the reduced echelon form

$$\begin{pmatrix} 1 & 0 & 7/5 & 1 & 7/5 \\ 0 & 1 & 1/5 & 2 & 11/5 \end{pmatrix} = (\, I \quad T\,).$$

Then the columns of the following matrix form a basis for the null space :

$$\begin{pmatrix} T \\ -I \end{pmatrix} = \begin{pmatrix} 7/5 & 1 & 7/5 \\ 1/5 & 2 & 11/5 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

6. Solve the following system of linear equations

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & & & = & 1 \\ -x_1 & - & 4x_2 & + & x_3 & = & 2 \\ & & 2x_2 & + & x_3 & = & 0. \end{array}$$

Solution $\quad (x_1, x_2, x_3) = (5/2,\ -3/4,\ 3/2)$

7. Determine whether the following system of linear equations has a solution or not.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 0 \end{pmatrix}.$$

Solution No solution exists.

8. Find a 2 by 2 square matrix B satisfying

$$A = B \cdot C,$$

where $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & -3 & 0 \\ 8 & 9 & 3 \end{pmatrix}$.

Solution

$$C = \begin{bmatrix} -1 & 0 \\ 2/3 & 1/3 \end{bmatrix}$$

9. Suppose the matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is such that $A \in \mathbb{R}^{a\times 3}$, $B \in \mathbb{R}^{2\times b}$, $C \in \mathbb{R}^{c\times d}$, and $D \in \mathbb{R}^{5\times 4}$.

(a) What are the values of $a, b, c,$ and d?

(b) Suppose that the matrix multiplication MT is well defined. Further suppose that it can be done in block form where T has the structure

$$T = \begin{bmatrix} U & V & W \\ Q & R & S \end{bmatrix}.$$

What are the possible dimensions of the matrices $U, V, W, Q, R,$ and S?

Solution (a) $(a, b, c, d) = (2, 4, 5, 3)$ (b) $U \in \mathbb{R}^{3\times k_1}$, $V \in \mathbb{R}^{3\times k_2}$, $W \in \mathbb{R}^{3\times k_3}$, $Q \in \mathbb{R}^{4\times k_1}$, $R \in \mathbb{R}^{4\times k_2}$, $S \in \mathbb{R}^{4\times k_3}$ with k_1, k_2, k_3 arbitrary positive integers.

10. Consider the matrix

$$\begin{bmatrix} 2 & 1 & 3 & 4 & 9 \\ 2 & -2 & -4 & 2 & 8 \\ 4 & -1 & 2 & 1 & 7 \\ 1 & 1 & 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

If $A \in \mathbb{R}^{a\times 2}$ and $D \in \mathbb{R}^{2\times d}$, determine a and d then compute the matrix product CB.

Solution

$$CB = \begin{pmatrix} 16 & 14 & 28 \\ -1 & 6 & 17 \end{pmatrix}.$$