T53

Complex Eigenvalues

<u>Th</u>: If $\lambda \in \mathbb{C}$ is a complex eigenvalue of the real matrix $A \in \mathbb{R}^{n \times n}$ with eigenvector $x \in \mathbb{C}^n$, then $\bar{\lambda}$ is also an eigenvalue of A with eigenvector $\bar{x}$.

<u>Pf</u>: $Ax = \lambda x \quad \Rightarrow$

$$A\bar{x} = \bar{A}\bar{x} = \overline{(Ax)} = \overline{(\lambda x)} = \bar{\lambda}\,\bar{x}.$$

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Fundamental Theorem of Algebra

Let $p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$ with $a_n \neq 0$.

Then p has n roots in $\mathbb{C}$ counting multiplicity.

If p is real, i.e. $a_0, a_1, \ldots, a_n \in \mathbb{R}$, and $\lambda = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$ is a complex root of p with $\beta \neq 0$ ~~if and only if~~ Then $\bar{\lambda} = \alpha - i\beta$ is also a root of p.

Consequently, if p is a real polynomial then there exist non-negative integers s and t with $n = s + 2t$ such that

$$p(\lambda) = a_n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_s) \cdot (\lambda^2 - 2\alpha_1\lambda + \alpha_1^2 + \beta_1^2)(\lambda^2 - 2\alpha_2\lambda + \alpha_2^2 + \beta_2^2)\cdots(\lambda^2 - 2\alpha_t\lambda + \alpha_t^2 + \beta_t^2)$$

where $\underbrace{\lambda_1, \lambda_2, \ldots, \lambda_s}_{\in \mathbb{R}}$, $\underbrace{\alpha_1 \pm i\beta_1, \ldots, \alpha_s \pm i\beta_s}_{\in \mathbb{C}}$ are the roots of p.

$\alpha_i, \beta_i \in \mathbb{R}$

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$$P(\lambda) = a\lambda^2 + b\lambda + c$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ex: $A = \begin{bmatrix} -1 & 3 & -4 \\ -2 & 3 & -4 \\ 1 & 1 & 3 \end{bmatrix}$

$$|A - \lambda I| = -(\lambda - 1)(\lambda^2 - 4\lambda + 13)$$

roots are $\lambda = 1$, $\lambda = 2 \pm 3i$

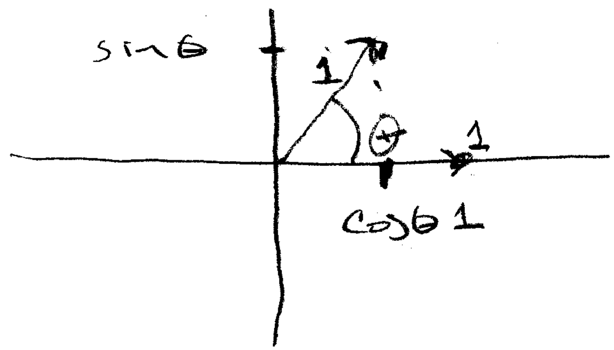
156 Rotation Matrices

Suppose we wish to rotate the vectors in $\mathbb{R}^2$ through an angle of θ radians. This turns out to be a linear transformation. The matrix associated with this linear transformation is given by its columns which are determined by how the transformation rotates the vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$,

To express this transformation we use polar coordinates. $x = r\cos\theta$ and $y = r\sin\theta$

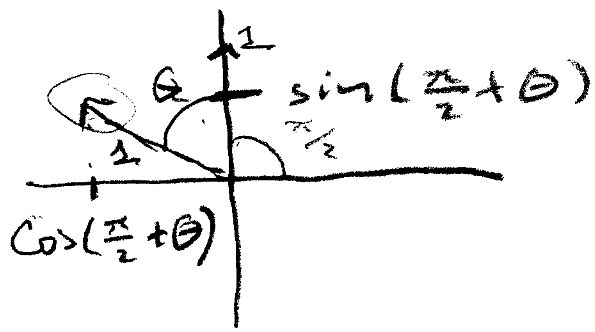
recall that
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

(157) rotate the vector $\begin{pmatrix}1\\0\end{pmatrix}$ through an angle θ



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

rotate the vector $\begin{pmatrix}0\\1\end{pmatrix}$ through an angle θ



$\frac{\pi}{2}+\theta$

$$\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}\cos(\frac{\pi}{2}+\theta)\\ \sin(\frac{\pi}{2}+\theta)\end{pmatrix} = \begin{pmatrix}\cos\frac{\pi}{2}\cos\theta - \sin\frac{\pi}{2}\sin\theta\\ \sin\frac{\pi}{2}\cos\theta + \cos\frac{\pi}{2}\sin\theta\end{pmatrix}$$

$$= \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

158 Therefore the matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ (Givens Rotation)

rotates the xy-plane counter clockwise about the origin through an angle θ.

What are the eigenvalues of R_θ?

$$\begin{vmatrix} c-\lambda & -s \\ s & c-\lambda \end{vmatrix} = (c-\lambda)^2 + s^2 = \lambda^2 - 2c\lambda + c^2 + s^2$$

$$= (\lambda - (c+is))(\lambda - (c-is))$$

$$\lambda = c \pm is$$

159 Rotation-Dilation Matrices

$$\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix}$$

$$R_{(r,\theta)} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$|R_{(r,\theta)} - \lambda I| = \begin{vmatrix} rc-\lambda & -rs \\ rs & rc-\lambda \end{vmatrix} = (rc-\lambda)^2 + r^2s^2$$

$$= \lambda^2 - 2rc\lambda + r^2s^2 + r^2c^2$$

$$= (\lambda - (rc + irs))(\lambda - (rc - irs))$$

magnitude or modulus

$$r = |a \pm ib| = (a^2+b^2)^{1/2}$$

$$\lambda = r(c \pm is) = a \pm ib$$

$$\theta = \sin^{-1}\left(\frac{b}{r}\right)$$

argument

$Re(a+ib) = a$, $Im(a+ib) = b$

(59.5) Recognizing Rotation - Dilation Matrices

Any real matrix of the form

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad \left(\begin{matrix} \text{skew symmetric} \\ A^T = -A \end{matrix} \right)$$

is a rotation-dilation matrix.

$$\begin{aligned} r\cos\theta &= a \\ r\sin\theta &= b \end{aligned} \implies r^2 = a^2+b^2 \qquad \theta = \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) = \cos^{-1}\left(\frac{a}{r}\right)$$

Note $|A - \lambda I| = (a-\lambda)^2 + b^2 = \lambda^2 - 2a\lambda + a^2 + b^2$.

$\Rightarrow \lambda = a \pm ib$

$\theta = \arg(a+ib) \qquad r = \sqrt{a^2+b^2}$

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ex: $A = \begin{bmatrix} 3 & -4 \\ 1 & 3 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -4 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 4$$

$$0 = (3-\lambda)^2 + 4 \implies (\lambda - 3)^2 = -4$$

$$\implies \lambda - 3 = \pm\sqrt{-4} = \pm 2i$$

$$\implies \lambda = 3 \pm 2i$$

$$A - \lambda I = \begin{bmatrix} 3-(3-2i) & -4 \\ 1 & 3-(3-2i) \end{bmatrix} = \begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix}$$

solve the homogeneous equation $\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix} \to \begin{bmatrix} 1 & 2i \\ 0 & 0 \end{bmatrix}$ eigen vector $\begin{bmatrix} 2i \\ -1 \end{bmatrix} = i\begin{pmatrix} 2 \\ i \end{pmatrix}$

so A has complex eigenvector $u = \begin{pmatrix} 2 \\ i \end{pmatrix}$

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$$Ax = \begin{bmatrix} 3 & -4 \\ 1 & 3 \end{bmatrix}\begin{pmatrix} 2 \\ i \end{pmatrix} = \begin{bmatrix} 6-4i \\ 2+3i \end{bmatrix} = (3-2i)\begin{pmatrix} 2 \\ i \end{pmatrix}$$

Let $P = [\mathrm{Re}(u), \mathrm{Im}(u)] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$= P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 6 & -4 \\ 2 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & -4 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} = \sqrt{3^2+2^2}\begin{bmatrix} \frac{3}{\sqrt{3^2+2^2}} & -\frac{2}{\sqrt{3^2+2^2}} \\ \frac{2}{\sqrt{3^2+2^2}} & \frac{3}{\sqrt{3^2+2^2}} \end{bmatrix}$$

$$= r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$r = \sqrt{13}$

$\theta = \cos^{-1}\left(\frac{3}{\sqrt{3^2+2^2}}\right)$

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<u>Theorem</u>: If $A \in \mathbb{R}^{2\times 2}$ has complex eigenvalue $\lambda = a - ib$ with associated eigenvector $u = \mathrm{Re}(u) + i\,\mathrm{Im}(u)$, then

$$A = PBP^{-1}$$

where $P = [\,\mathrm{Re}(u) \;\; \mathrm{Im}(u)\,]$ and

$B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is a rotation-dilation matrix

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Similarity Transformations

Let $A \in \mathbb{R}^{n\times n}$. A similarity transformation of A is a mapping of $\mathbb{R}^{n\times n}$ to itself given by

$$B = P^{-1} A P$$

where $P \in \mathbb{R}^{n\times n}$ is invertible. In this case we say that B is similar to A.

<u>Theorem</u>: Let $A \in \mathbb{R}^{n\times n}$ have eigenvalue λ with an associated eigenvector x. Let $P \in \mathbb{R}^{n\times n}$ be invertible and set $B = P^{-1}AP$. Then $y = P^{-1}x$ is an eigenvector of B with associated eigenvalue λ.

<u>Pf</u>: First note that since $B = P^{-1}AP$ we have $BP^{-1} = P^{-1}A$.
Hence $By = BP^{-1}x = P^{-1}Ax = \lambda P^{-1}x = \lambda y$.