R1 Gram-Schmidt Orthogonalization Process 12/01/21

Given a set of linearly independent vectors

$$\{x_1, x_2, \dots, x_k\} \subset \mathbb{R}^n$$

Find a set of vectors $\{u_1, u_2, \dots u_k\}$ such that

$$u_i^T u_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and $\text{Span}[u_1, u_2, \dots, u_k] = \text{Span}[x_1, x_2, \dots, x_k]$.

such a set of vectors is called an orthonormal basis for $\text{Span}[x_1, x_2, \dots, x_k]$.

Note: If $U = [u_1\ u_2 \dots u_k] \in \mathbb{R}^{n \times k}$, then

$$U^T U = I_k$$

$k = n$

R2

12/06/21

$\{x_1, x_2\}$ linearly independent

$y_1 = x_1$

$u_1 = \frac{y_1}{\|y_1\|}$

$y_2 = x_2 - (u_1^T x_2) u_1$

$u_2 = \frac{y_2}{\|y_2\|}$

$\Rightarrow \quad 1 = u_1^T u_1 = u_2^T u_2$

$$u_1^T u_2 = \frac{1}{\|y_2\|} u_1^T [x_2 - (u_1^T x_2) u_1]$$

$$= \frac{1}{\|y_2\|} [u_1^T x_2 - (u_1^T x_2) u_1^T u_1]$$

$$= \frac{1}{\|y_2\|} [u_1^T x_2 - u_1^T x_2]$$

$$= 0$$

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<u>$\mathbb{R}_3$</u>

Note That

$$y_2 = x_2 - (u_1^T x_2) u_1$$

$$= x_2 - u_1 u_1^T x_2$$

$$= [I - u_1 u_1^T] x_2$$

where

$$[I - u_1 u_1^T]^T = [I - u_1 u_1^T]$$

$$[I - u_1 u_1^T] u_1 = u_1 - u_1 (u_1^T u_1) = u_1 - u_1 = 0$$

$$[I - u_1 u_1^T]^2 = [I - u_1 u_1^T][I - u_1 u_1^T] = I - u_1 u_1^T - u_1 u_1^T + u_1 u_1^T u_1 u_1^T]$$

$$= I - u_1 u_1^T$$

$\text{nullity}([I - u_1 u_1^T]) = 1$, $\text{rank}(I - u_1 u_1^T) = n-1$

R4

12/01/21

$\{x_1, x_2, x_3\}$ linearly independent

$y_1 = x_1$

$u_1 = \dfrac{y_1}{\|y_1\|}$

$y_2 = x_2 - (u_1^T x_2) u_1 = [I - u_1 u_1^T] x_2$

$u_2 = \dfrac{y_2}{\|y_2\|}$

$y_3 = x_3 - (u_1^T x_3) u_1 - (u_2^T x_3) u_2 = \{I - u_1 u_1^T - u_2 u_2^T\} x_3$

$u_3 = \dfrac{y_3}{\|y_3\|}$

We have $u_1^T u_1 = u_2^T u_2 = u_3^T u_3 = 0$, $u_1^T u_2 = 0$

$$u_1^T u_3 = u_1^T \frac{[I - u_1 u_1^T - u_2 u_2^T] x_3}{\|y_3\|} = \frac{1}{\|y_2\|} \underbrace{[u_1^T - (\overset{=1}{u_1^T u_1}) u_1^T - (\overset{=0}{u_1^T u_2}) u_2^T]}_{=0} x_3 = 0$$

$$u_2^T u_3 = \frac{1}{\|y_3\|} u_2^T \{I - u_1 u_1^T - u_2 u_2^T\} x_3 = \frac{1}{\|y_3\|} \underbrace{[u_2^T - \underset{0}{u_2^T u_1} u_1^T - \underset{1}{u_2^T u_2} u_2^T]}_{=0} x_3 = 0$$

12/01/21

R5 $\{x_1, x_2, \dots x_k\}$ linearly independent

$$Y_1 = x_1$$

$$u_1 = \frac{Y_1}{\|Y_1\|}$$

$$Y_2 = x_2 - (u_1^T x_2) u_1$$

$$u_2 = \frac{Y_2}{\|Y_2\|}$$

$$Y_3 = x_3 - (u_1^T x_3) u_1 - (u_2^T x_3) u_2$$

$$u_2 = \frac{Y_3}{\|Y_3\|}$$

$$\vdots$$

$$Y_k = x_k - (u_1^T x_k) u_1 - (u_2^T x_k) u_2 - \dots - (u_{k-1}^T x_k) u_{k-1}$$

$$= [I - u_1 u_1^T - u_2 u_2^T - u_3 u_3^T - \dots - u_{k-1} u_{k-1}^T] x_k$$

$$u_k = \frac{Y_k}{\|Y_k\|}$$

R6 12/01/21

We have $u_i^T u_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

for $i = j = 1, 2, \ldots, k$

$i \neq j$, $i = 1 \ldots k-1$
$j = 1 \ldots k-1$

Consider $u_j^T u_k$ for $j = 1, 2, \ldots, k-1$

$$u_j^T u_k = \frac{1}{\|y_k\|} u_j^T \{ I - u_1 u_1^T - u_2 u_2^T - \ldots - u_{k-1} u_{k-1}^T \} x_k$$

$$= \frac{1}{\|y_k\|} [u_j^T - (u_j^T u_1) u_1^T - (u_j^T u_2) u_2^T - \ldots - (u_j^T u_{k-1}) u_{k-1}^T] x_k$$

$$= \frac{1}{\|y_k\|} [u_j^T - (u_j^T u_j) u_j^T] x_k$$

$$= \frac{1}{\|y_k\|} [u_j^T - u_j^T] x_k$$

$= 0$ $\quad j = 1, 2, \ldots, k-1$

(145) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ Compute an orthonormal basis for Ran(A).

$$y_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad u_1 = \frac{y_1}{\|y_1\|} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$y_2 = x_2 - (u_1^T x_2) u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 1 \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$u_2 = \frac{y_2}{\|y_2\|} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$y_3 = x_3 - (u_1^T x_3) u_1 - (u_2^T x_3) u_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 0 \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

so $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ is an orthonormal basis for Ran(A).

(146) ex: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$y_1 = x_1$

$u_1 = \dfrac{y_1}{\|y_1\|} = \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$y_2 = x_2 - (u_1^T x_2)\, u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \dfrac{2}{\sqrt{3}} \cdot \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \dfrac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \dfrac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$u_2 = \dfrac{y_2}{\|y_2\|} = \dfrac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$y_3 = x_3 - (u_1^T x_3) u_1 - (u_2^T x_3) u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \dfrac{2}{\sqrt{3}} \cdot \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \dfrac{-1}{\sqrt{6}} \dfrac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \dfrac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \dfrac{1}{6} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \dfrac{1}{6} \begin{bmatrix} 6 & -4 & +1 \\ 6 & -4 & -2 \\ & -4 & +1 \end{bmatrix} = \dfrac{1}{6} \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} = \dfrac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$u_3 = \dfrac{y_2}{\|y_2\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

146.5

$$A = \begin{bmatrix} 2 & -5 & 6 \\ -1 & 2 & -3 \\ 3 & 1 & 4 \\ 0 & -1 & 6 \end{bmatrix}$$

$$Y_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 0 \end{pmatrix}$$

$$u_1 = \frac{Y_1}{\|Y_1\|} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -1 \\ 3 \\ 0 \end{pmatrix}$$

$$Y_2 = X_2 - (u_1^T X_2) u_1 = \begin{pmatrix} -5 \\ 2 \\ 1 \\ -1 \end{pmatrix} - \frac{-9}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 1 \\ -1 \end{pmatrix} + \frac{9}{14} \begin{pmatrix} 2 \\ -1 \\ 3 \\ 0 \end{pmatrix}$$

$$= \frac{1}{14} \left[\begin{pmatrix} -70 \\ 28 \\ 14 \\ -14 \end{pmatrix} + \begin{pmatrix} 18 \\ -9 \\ 27 \\ 0 \end{pmatrix} \right] = \frac{1}{14} \begin{pmatrix} -52 \\ 19 \\ 41 \\ -14 \end{pmatrix}$$

$$\|Y_2\| = \left[\frac{1}{196} (2704 + 361 + 1681 + 196) \right]^{1/2} = \left[\frac{4,942}{196} \right]^{1/2}$$

$$u_2 = \frac{1}{\sqrt{4,942}} \begin{pmatrix} -52 \\ 19 \\ 41 \\ -14 \end{pmatrix}$$

. . . . .

146.6

$\mathbb{C}$ ~ complex numbers $\xi = \alpha + i\beta$ $\quad \alpha, \beta \in \mathbb{R}$

$i = \sqrt{-1}$

$$\mathbb{C}^n = \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} : z_i \in \mathbb{C} \quad i = 1 \ldots n \right\}$$

<u>Conjugates</u> $\xi = \alpha + i\beta \quad , \bar{\xi} = \alpha - i\beta$

<u>mult</u> $(\alpha + i\beta)(\gamma + i\delta) = (\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma)$

$= (\gamma + i\delta)(\alpha + i\beta)$ (commutative)

<u>modulus</u> $\xi = \alpha + i\beta \quad , \bar{\xi}\xi = (\alpha - i\beta)(\alpha + i\beta) = \alpha^2 + \beta^2 = |\xi|^2$

$|\xi| = \sqrt{\bar{\xi}\xi} \quad , \overline{\xi\zeta} = \bar{\xi}\,\bar{\zeta} \quad , \xi, \zeta \in \mathbb{C}$

<u>Vectors</u>: $z \in \mathbb{C}^n$, $\bar{z} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{pmatrix}$, $\bar{z}^T z = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$

$\|z\| = \sqrt{\bar{z}^T z}$

(147) Symmetric Matrices

$A \in \mathbb{R}^{n \times n}$ is said to be symmetric if $A = A^T$.

$\mathcal{S}^n$

<u>eg</u>: $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Eigenvalues of Symmetric Matrices must be real.

Let $A \in \mathbb{R}^{n \times n}$ be symmetric and let λ be an eigenvalue of A with associated eigenvector $0 \neq w \in \mathbb{C}^n$, i.e. $Aw = \lambda w$.

Conjugating yields $A\bar{w} = \overline{\lambda w} = \bar{\lambda}\bar{w}$

$\Rightarrow \bar{\lambda} \underline{w^T \bar{w}} = w^T A \bar{w} = \bar{w}^T A^T x = \bar{w}^T A x = \lambda \underline{\bar{w}^T w}$

$\underline{\bar{w}^T w} = \sum_{i=1}^n \bar{w}_i w_i = \underline{\sum_{i=1}^n |w_i|^2} > 0$

$\Rightarrow \bar{\lambda} = \lambda$

$\Rightarrow \lambda$ is real.

(148) Let $A \in \mathbb{R}^{n \times n}$ be symmetric and let λ be an eigenvalue of A.

We have shown that $\lambda \in \mathbb{R}$.

Now suppose $w \in \mathbb{C}^n$ is an eigenvector of A associated with λ, so that $Aw = \lambda w$.

If $w = x + iy$ with $x, y \in \mathbb{R}^n$, then

$$Ax + iAy = Aw = \lambda w = \lambda x + i(\lambda y)$$

so $Ax = \lambda x$ and $Ay = \lambda y$.

That is, both the real and complex parts of w are eigenvectors of A associated with λ.

In general, $\mathrm{Nul}(A - \lambda I) \neq \{0\}$ so there must be a real eigenvector associated with λ.

(149) <u>Thm</u>: Let $A \in \mathbb{R}^{n\times n}$ be symmetric

Then the eigenvectors of A associated with distinct eigenvalues are orthogonal.

<u>pf</u>: λ_i is an eigenvalue of A associated with the eigenvector x_i, $i = 1, 2$.

Suppose $\lambda_1 \neq \lambda_2$.

Then

$$\lambda_1 x_2^T x_1 = x_2^T(\lambda_1 x_1) = x_2^T A x_1 = x_1^T A x_2 = x_1^T(\lambda_2 x_2) = \lambda_2 x_2^T x_1$$

$\Rightarrow (\lambda_1 - \lambda_2) x_2^T x_1 = 0$

$\Rightarrow x_2^T x_1 = 0$ so x_1 and x_2 are orthogonal.

(150) Unitary Matrices

A matrix $U \in \mathbb{R}^{n \times n}$ is said to be unitary if $U^T U = I_n$, i.e. $U^{-1} = U^T$.

Thm: If $U = [u_1\ u_2 \cdots u_n] \in \mathbb{R}^n$ is unitary if and only if $\{u_1, u_2, \ldots, u_n\}$ is an orthonormal basis of $\mathbb{R}^n$.

Pf: $I = U^T U = \{ u_i^T u_j \} \iff u_i^T u_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$\iff \{u_1, \ldots, u_n\}$ form an orthonormal basis.

(151)

Lemma: Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Let $\lambda \in \mathbb{R}$ be an eigenvalue of A with eigenvector u, $\|u\| = 1$

Then there exists a matrix $\hat{W} \in \mathbb{R}^{n\times(n-1)}$ s.t. $W = [u \ \hat{W}]$

is unitary and

$$W^T A W = \begin{bmatrix} \lambda & 0 \\ 0 & \hat{W}^T A \hat{W} \end{bmatrix}.$$

pf: Choose $w_2, w_3, \dots, w_n \in \mathbb{R}^n$ s.t. $\{u, w_2, \dots, w_n\}$ form an orthonormal basis for $\mathbb{R}^n$.

Set $\hat{W} = [w_2 \ w_3 \ \dots \ w_n] \in \mathbb{R}^{n\times(n-1)}$, $W = [u \ \hat{W}]$

Then W is unitary and

$$W^T A W = \begin{bmatrix} u^T \\ \hat{W}^T \end{bmatrix} A [u \ \hat{W}] = \begin{bmatrix} u^T \\ \hat{W}^T \end{bmatrix} [Au \ A\hat{W}]$$

$$= \begin{bmatrix} u^T A u & u^T A \hat{W} \\ \hat{W}^T A u & \hat{W}^T A \hat{W} \end{bmatrix} = \begin{bmatrix} \lambda & \lambda u^T \hat{W} \\ \lambda \hat{W}^T u & \hat{W}^T A \hat{W} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \hat{W}^T A \hat{W} \end{bmatrix}$$

152 **Theorem**: Let $A \in \mathbb{R}^{n \times n}$ be symetric.
Then A is unitarily diagonalizable, i.e

$$A = U \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} U^T$$

where $U^T U = I$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.