R1 11/29/21

Ex: $A = \begin{bmatrix} 3 & -5 & 3 & 2 & -3 \\ 5 & -7 & 3 & 1 & 1 \\ 6 & -6 & 2 & -1 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

$\left| \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = (1-\lambda)(2-\lambda)$

$$\begin{vmatrix} 3-\lambda & -5 & 3 \\ 5 & -7-\lambda & 3 \\ 6 & -6 & 2-\lambda \end{vmatrix} = 3 \begin{vmatrix} 5 & -(7+\lambda) \\ 6 & -6 \end{vmatrix} - 3 \begin{vmatrix} 3-\lambda & -5 \\ 6 & -6 \end{vmatrix} + (2-\lambda) \begin{vmatrix} 3-\lambda & -5 \\ 5 & -(7+\lambda) \end{vmatrix}$$

$$= 3[-30 + 6(7+\lambda)] - 3[-6(3-\lambda) + 30] + (2-\lambda)[-(3-\lambda)(7+\lambda) + 25]$$

$$= 3[12 + 6\lambda] - 3[12 + 6\lambda] + (2-\lambda)[\lambda^2 + 4\lambda + 4]$$

$$= (2-\lambda)(\lambda+2)^2 \quad , \quad p(\lambda) = (1-\lambda)(2-\lambda)^2(2+\lambda)^2$$

eigenvalues: $\lambda = 2, -2, 1$

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P2

$\underline{\lambda = 1}$

$$\begin{array}{ccccc} 2 & -5 & 3 & 2 & -3 \\ 5 & -8 & 3 & 1 & 1 \\ 6 & -6 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 2 & -2 & -2 & 0 \\ 0 & -9 & 7 & 6 & 0 \\ 0 & -18 & 13 & 11 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 2 & -2 & -2 & 0 \\ 0 & 9 & -7 & -6 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 9 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{ccccc} 1 & 0 & 0 & -2/9 & 0 \\ 0 & 1 & 0 & 1/9 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$$x = \begin{pmatrix} -2/9 \\ 1/9 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -2 \\ 1 \\ 9 \\ -9 \\ 0 \end{pmatrix}$$

<u>check</u>: $-4 - 5 + 27 - 18 = 0$

$-10 - 8 + 27 - 9 = 0$

$-12 - 6 + 9 + 9 = 0$

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R3

$\lambda = 2$

$$\begin{matrix} 1 & -5 & 3 & 2 & -3 \\ 5 & -9 & 3 & 1 & 1 \\ 6 & -6 & 0 & -1 & 5 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & -5 & 3 & 0 & 3 \\ 5 & -9 & 3 & 0 & 4 \\ 6 & -6 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 16 & -12 & 0 & -11 \\ 0 & 8 & -6 & 0 & -5 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 8 & -6 & 0 & 5 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \quad \begin{matrix} 1 & 0 & -3/4 & 0 & 0 \\ 0 & 1 & -3/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$x = \begin{pmatrix} -3/4 \\ -3/4 \\ -1 \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 3 \\ 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

Check: $3 - 15 + 12 = 0$

$15 - 27 + 12 = 0$

$18 - 18 = 0$

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$\lambda = -2$

$$\begin{matrix} 5 & -5 & 3 & 2 & -3 \\ 5 & -5 & 3 & 1 & 1 \\ 6 & -6 & 4 & -1 & 5 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{matrix}$$

$$x = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \checkmark$$

$$\begin{matrix} 5 & -5 & 3 & 2 & 0 \\ 5 & -5 & 3 & 1 & 0 \\ 6 & -6 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 5 & -5 & 3 & 0 & 0 \\ 5 & -5 & 3 & 0 & 0 \\ 6 & -6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

12 5

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eigenvector eigenvalue pairs

$\lambda = 1, \begin{pmatrix} -2 \\ 1 \\ 9 \\ -9 \\ 0 \end{pmatrix}, \lambda = 2, x = \begin{pmatrix} 3 \\ 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}, \lambda = -2, x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

A = is not diagonalizable.

Definition: Given $A \in \mathbb{R}^{n \times n}$ the characteristic polynomial for A is given by

$$p(\lambda) = \det(A - \lambda I)$$

$p(\lambda)$ is a polynomial of degree n.

Two step process for finding eigenvector eigenvalue pairs

1. Solve the degree n polynomial equation
   $p(\lambda) = \det(A - \lambda I) = 0$.
   The roots of p are the eigenvalues of A.
   There are n such roots counting multiplicity $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$

2. Solve the homogeneous equation $(A - \lambda_i I)x = 0$
   for the eigenvectors of A associated
   with the eigenvalue λ_i, $i = 1, \ldots, n$.

(131) $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

$|A - \lambda I| = \begin{bmatrix} -\lambda & -1 & 0 \\ 0 & 1-\lambda & 0 \\ -1 & -1 & 1-\lambda \end{bmatrix} = (1-\lambda)(-1)^{3+3} \begin{vmatrix} -\lambda & -1 \\ 0 & 1-\lambda \end{vmatrix}$

$= -\lambda(1-\lambda)^2 \quad , \lambda = 0, 1, 1$

$\lambda = 0, \quad x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1 \quad \left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right]$

$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(132) $A = \begin{bmatrix} -5 & -10 & 6 \\ 6 & 11 & -6 \\ 5 & 8 & -4 \end{bmatrix}$

$\begin{array}{r} 8 \\ +8 \\ -22 \end{array}$

$|A - \lambda I| = \begin{vmatrix} -5-\lambda & -10 & 6 \\ 6 & 11-\lambda & -6 \\ 5 & 8 & -4-\lambda \end{vmatrix} = 6(-1)^{3+1} \begin{vmatrix} 6 & 11-\lambda \\ 5 & 8 \end{vmatrix} + 6 \begin{vmatrix} -(5+\lambda) & -10 \\ 5 & 8 \end{vmatrix}$

$- (4+\lambda)(-1)^{3+3} \begin{vmatrix} -(5+\lambda) & -10 \\ 6 & 11-\lambda \end{vmatrix}$

$= 6[48 - 5(11-\lambda)] + 6[50 - 8(5+\lambda)] - (4+\lambda)[60 - (5+\lambda)(11-\lambda)]$

$= 6[3 - 3\lambda] - (4+\lambda)[60 - 55 - 6\lambda + \lambda^2]$

$= 18(1-\lambda) - (4+\lambda)(\lambda^2 - 6\lambda + 5)$

$= 18 - 18\lambda - [\lambda^3 - 6\lambda^2 + 5\lambda + 4\lambda^2 - 24\lambda + 20]$

$= -\lambda^3 + 2\lambda^2 + \lambda - 2$

$\lambda = 1, 2, -1$

$0 = \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda-1)(\lambda-2)(\lambda+1)$

$$\begin{array}{r l} & \lambda^2 - \lambda - 2 \\ \lambda - 1 \,\big)\!\!\! & \overline{\lambda^3 - 2\lambda^2 - \lambda + 2} \\ & \underline{\lambda^3 - \lambda^2} \\ & 0 \quad -\lambda^2 - \lambda + 2 \\ & \quad\; \underline{-\lambda^2 + \lambda} \\ & \quad\; 0 \quad -2\lambda + 2 \\ & \qquad\quad -2\lambda + 2 \end{array}$$

$\lambda = \dfrac{1 \pm \sqrt{1+8}}{2}$

$= \dfrac{1 \pm 3}{2}$

$= 2, -1$

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If λ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ of multiplicity k, then $\dim(\text{Nul}(A-\lambda I)) \le k$.

algebraic

ex: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$

solve $(A-I)x=0$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the only solution

(134) The Unifying Theorem Version 8

Let $S = \{a^1, a^2, \ldots, a^n\} \subset \mathbb{R}^n$, $A = [a^1 \ldots a^n]$, $T_A x = Ax$

TFAE

(a) $\operatorname{span} S = \mathbb{R}^n$

(b) S is linearly independent

(c) $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$

(d) T_A is onto

(e) T_A is one to one

(f) A is invertible

(g) $\ker(T) = \{0\}$

(h) S is a basis for $\mathbb{R}^n$

(i) $\operatorname{col}(A) = \operatorname{Ran}(A) = \mathbb{R}^n$

(k) $\operatorname{row}(A) = \operatorname{Ran}(A^T) = \mathbb{R}^n$

(l) $\det(A) \neq 0$

(m) $\operatorname{Nul}(A) = \{0\}$

(n) $\lambda = 0$ is not an eigenvalue of A

(m) $\lambda = 0$

(135) 6.2 Diagonalization :

$x \in \mathbb{R}^n$, $D = \text{Diag}[x] = \begin{bmatrix} x_1 & \cdots & & 0 \\ 0 & \ddots & 0 & \vdots \\ & 0 & \ddots & 0 \\ 0 & \cdots & 0 & x_n \end{bmatrix} \Rightarrow$ x_i is an eigenvalue of D with eigenvector e_i

<u>Defn</u>: We say that $A \in \mathbb{R}^{n \times n}$ is diagonalizable if there exists $P \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$ such that P is invertible and $A = P\,\text{Diag}(x)\,P^{-1}$

Observation: Suppose $A = P\,\text{Diag}(x)\,P^{-1}$.

$P = [p^1\ p^2 \ldots p^n]$. Then $P^{-1}P = [q_i^T p_j] = I$

$P^{-1} = \begin{bmatrix} q_1^T \\ q_n^T \end{bmatrix}$ $\qquad q_i^T p_j = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$

$\Rightarrow P^{-1}p^j = e_j$

$\Rightarrow Ap^j = P\,\text{Diag}(x)\,P^{-1}p^j = P\,\text{Diag}(x)\,e_j = x_j P e_j = x_j p^j$

$\Rightarrow p^j$ is the eigenvector associated with the eigenvalue x_j

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ex: $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -\lambda & 1 \end{vmatrix}$$

$$= -\lambda\{-\lambda(2-\lambda)+1\} - \{2-\lambda-1\} + \{1-\lambda\}$$

$$= -\lambda\{-2\lambda+\lambda^2+1\}$$

$$= -\lambda\{\lambda^2-2\lambda+1\} = -\lambda(\lambda-1)^2$$

$\lambda = 0, 1, 1$

$\lambda = 0$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 1$

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

check

$P = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

137 $P = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

check

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 2 \end{array}\right]$$

$$P \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} P^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} = A$$

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<u>Theorem</u>: $A \in \mathbb{R}^{n \times n}$ is diagonalizable $\iff$ eigenvectors form a basis for $\mathbb{R}^n$.

Pf: ($\Rightarrow$) $A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}$, set $P = [p^1\, p^2 \cdots p^n]$, $[P^{-1}]^T = [q^1\, q^2 \cdots q^n]$

$$\Rightarrow I_n = P^{-1}P = \begin{bmatrix} (q^1)^T \\ (q^2)^T \\ \vdots \\ (q^n)^T \end{bmatrix} [p^1\, p^2 \cdots p^n] = \begin{bmatrix} (q^1)^T p^1 & (q^1)^T p^2 & \cdots & (q^1)^T p^n \\ (q^2)^T p^1 & (q^2)^T p^2 & \cdots & (q^2)^T p^n \\ \vdots & & & \\ (q^n)^T p^1 & (q^n)^T p^2 & \cdots & (q^n)^T p^n \end{bmatrix}$$

$$\Rightarrow (q^i)^T p^j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$\Rightarrow P^{-1}p^j = e_j$ the j^{th} unit coordinate vector $\Rightarrow Ap^j = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}p^j = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} e_j$
$= \lambda_j P e_j = \lambda_j p^j$

$\Rightarrow p^j$ is an eigenvector of A with eigenvalue λ_j.

($\Leftarrow$) Write $P = [p^1 \cdots p^n]$ $\Rightarrow P^{-1}AP = P^{-1}[Ap^1 \cdots Ap^n] = P^{-1}[\lambda_1 p^1\, \lambda_2 p^2 \cdots \lambda_n p^n]$
$= [\lambda_1 P^{-1}p^1 \cdots \lambda_n P^{-1}p^n] = P^{-1}P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$\Rightarrow A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}$

(139) Thm: If $\{\lambda_1, \ldots, \lambda_k\}$ is a set of distinct eigenvectors of A. Then the associated set of eigenvectors $\{u^1, u^2, \ldots, u^k\}$ are linearly independent.

Pf: Spps false $\Rightarrow$ wlog $u^1 = \alpha_2 u^2 + \ldots + \alpha_k u^k$, not all $\alpha_i = 0$

$\Rightarrow \lambda_1 u^1 = Au^1 = \alpha_2 \lambda_2 u^2 + \ldots + \alpha_k \lambda_k u^k$

$\Rightarrow u^1 = \alpha_2 \frac{\lambda_2}{\lambda_1} u^2 + \ldots + \alpha_k \frac{\lambda_k}{\lambda_1} u^k$

$\Rightarrow 0 = \beta_2 u^2 + \ldots + \beta_k u^k$, $\beta_j = \alpha_j \frac{\lambda_j}{\lambda_1}$

$\Rightarrow u^2 = \hat{\beta}_2 u^3 + \ldots + \hat{\beta} u^k$

$\vdots$

$u^i = \delta_i u^k \quad i = 1, \ldots, n$

$\Rightarrow u^{k-1} = \gamma u^k \quad \Rightarrow$

$\times$

140 **Th**: $A \in \mathbb{R}^n$ eigenvalues $\Lambda = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}^n$

Then A is diagonalizable $\iff \dim(\mathrm{Nul}(A - \lambda_i I)) = k_i$

where k_i = multiplicity of λ_i

in $p(\lambda) = \det(A - \lambda I)$.

Cor: If $A \in \mathbb{R}^{n \times n}$ has n distinct

eigenvalues, then A is diagonalizable.