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Thm: $A \in \mathbb{R}^{n \times n}$ has eigenvalue λ.
Then $\mathrm{Nul}(A - \lambda I)$ = set of all eigenvectors associated with λ.

Defn: If $A \in \mathbb{R}^{n \times n}$ has eigenvalue λ, we call $\mathrm{Nul}(A - \lambda I)$ the eigenspace of A associated with λ.

Thm: $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$

$$\mathrm{Nul}(A - \lambda I) \neq \{0\} \iff \det(A - \lambda I) = 0$$

$\iff \lambda$ is an eigenvalue of A

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ex: $A = \begin{bmatrix} 2 & 6 \\ 1 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 6 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6$$

$$= 2 - 3\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

$\lambda = 4, -1$

$A - 4I = \begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix}$ $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$A + I = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector associated with $\lambda = 4$

$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector associated with $\lambda = -1$

129 $A = \begin{bmatrix} 0 & -3 & -1 \\ -1 & 2 & 1 \\ 3 & -9 & -4 \end{bmatrix}$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -3 & -1 \\ -1 & 2-\lambda & 1 \\ 3 & -9 & -4-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 2-\lambda & 1 \\ -9 & -(4+\lambda) \end{vmatrix} + \begin{vmatrix} -3 & -1 \\ -9 & -(4+\lambda) \end{vmatrix} + 3 \begin{vmatrix} -3 & -1 \\ 2-\lambda & 1 \end{vmatrix}$$

$$= -\lambda[-(2-\lambda)(4+\lambda) + 9] + 3(4+\lambda) - 9 + 3[-3 + (2-\lambda)]$$

$$= \lambda(2-\lambda)(4+\lambda) - 9\lambda + \cancel{3\lambda} \cancel{+3} \cancel{-3} - \cancel{3\lambda}$$

$$= \lambda(8 - 2\lambda - \lambda^2) - 9\lambda$$

$$= -\lambda^3 - 2\lambda^2 + 8\lambda - 9\lambda = -\lambda^3 - 2\lambda^2 - \lambda = -\lambda(\lambda^2 + 2\lambda + 1)$$

$$= -\lambda(\lambda+1)(\lambda+1)$$

$\lambda = 0$

$$\begin{array}{ccc} 0 & -3 & -1 \\ -1 & 2 & 1 \\ 3 & -9 & -4 \\ \hline 1 & -2 & -1 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \\ \hline 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{array} \qquad \begin{pmatrix} 1/3 \\ \cancel{-1/3} \\ 1 \end{pmatrix}$$

$x = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

$\lambda = -1$

$$\begin{array}{ccc} 1 & -3 & -1 \\ -1 & 3 & 1 \\ 3 & -9 & -3 \\ \hline 1 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$$

$x = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

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Definition: Given $A \in \mathbb{R}^{n \times n}$ the characteristic polynomial for A is given by

$$p(\lambda) = \det(A - \lambda I)$$

$p(\lambda)$ is a polynomial of degree n.

Two step process for finding eigenvector eigenvalue pairs

(1) Solve the degree n polynomial equation

$$p(\lambda) = \det(A - \lambda I) = 0.$$

The roots of p are the eigenvalues of A.
There are n such roots counting multiplicity $\{\lambda_1, \lambda_2, \dots \lambda_n\}$

(2) Solve the homogeneous equation $(A - \lambda_i I)x = 0$ for the eigenvectors of A associated with the eigenvalue λ_i, $i = 1, \dots, n$.

(131) $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

$|A - \lambda I| = \begin{bmatrix} -\lambda & -1 & 0 \\ 0 & 1-\lambda & 0 \\ -1 & -1 & 1-\lambda \end{bmatrix} = (1-\lambda)(-1)^{3+3} \begin{vmatrix} -\lambda & -1 \\ 0 & 1-\lambda \end{vmatrix}$

$= -\lambda(1-\lambda)^2 \quad , \lambda = 0, 1, 1$

$\lambda = 0, \; x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1 \quad \left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right]$

$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \; x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(132) $A = \begin{bmatrix} -5 & -10 & 6 \\ 6 & 11 & -6 \\ 5 & 8 & -4 \end{bmatrix}$

$\begin{array}{r} 8 \\ +8 \\ -22 \end{array}$

$|A-\lambda I| = \begin{vmatrix} -5-\lambda & -10 & 6 \\ 6 & 11-\lambda & -6 \\ 5 & 8 & -4-\lambda \end{vmatrix} = 6(-1)^{3+1}\begin{vmatrix} 6 & 11-\lambda \\ 5 & 8 \end{vmatrix} + 6\begin{vmatrix} -(5+\lambda) & -10 \\ 5 & 8 \end{vmatrix}$

$- (4+\lambda)(-1)^{3+3}\begin{vmatrix} -(5+\lambda) & -10 \\ 6 & 11-\lambda \end{vmatrix}$

$= 6[48 - 5(11-\lambda)] + 6[50 - 8(5+\lambda)] - (4+\lambda)[60 - (5+\lambda)(11-\lambda)]$

$= 6\{3 - 3\lambda] - (4+\lambda)[60 - 55 - 6\lambda + \lambda^2]$

$= 18(1-\lambda) - (4+\lambda)(\lambda^2 - 6\lambda + 5)$

$= 18 - 18\lambda - [\lambda^3 - 6\lambda^2 + 5\lambda + 4\lambda^2 - 24\lambda + 20]$

$= -\lambda^3 + 2\lambda^2 + \lambda - 2$

$\lambda = 1, 2, -1$

$0 = \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda-1)(\lambda-2)(\lambda+1)$

$$\begin{array}{r l} & \lambda^2 - \lambda - 2 \\ \lambda - 1 \,\big) & \lambda^3 - 2\lambda^2 - \lambda + 2 \\ & \underline{\lambda^3 - \lambda^2} \\ & 0 \quad -\lambda^2 - \lambda + 2 \\ & \underline{\quad -\lambda^2 + \lambda} \\ & \quad 0 \quad -2\lambda + 2 \\ & \quad\quad -2\lambda + 2 \end{array}$$

$\lambda = \dfrac{1 \pm \sqrt{1+8}}{2}$

$= \dfrac{1 \pm 3}{2}$

$= 2, -1$

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If λ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$ of multiplicity k, then $\dim(\mathrm{Nul}(A-\lambda I)) \le k$.

ex: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$

solve $(A-I)x=0$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the only solution

(134) The Unifying Theorem Version 8

Let $S = \{a^1, a^2, \dots, a^n\} \subset \mathbb{R}^n$, $A = [a^1 \dots a^n]$ $T_A x = Ax$

TFAE

(a) $\operatorname{span} S = \mathbb{R}^n$

(b) S is linearly independent

(c) $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$

(d) T_A is onto

(e) T_A is one to one

(f) A is invertible

(g) $\ker(T) = \{0\}$

(h) S is a basis for $\mathbb{R}^n$

(j) $\operatorname{col}(A) = \operatorname{Ran}(A) = \mathbb{R}^n$

(k) $\operatorname{row}(A) = \operatorname{Ran}(A^T) = \mathbb{R}^n$

(l) $\det(A) \neq 0$

(m) $\lambda = 0$

(m) $\operatorname{Nul}(A) = \{0\}$

(n) $\lambda = 0$ is not an eigenvalue of A

6.2 Diagonalization:

$x \in \mathbb{R}^n_1$, $D = \text{Diag}[x] = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & x_n \end{bmatrix} \Rightarrow$ x_i is an eigenvalue of D with eigenvector e_i

<u>Defn</u>: We say that $A \in \mathbb{R}^{n\times n}$ is diagonalizable if there exists $P \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ such that P is invertible and $A = P\,\text{Diag}(x)\,P^{-1}$

Observation: Suppose $A = P\,\text{Diag}(x)\,P^{-1}$.

$P = [p^1\ p^2 \cdots p^n]$. Then $P^{-1}P = [q_i^T p_j] = I$

$P^{-1} = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$ $\quad q_i^T p_j = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$

$\Rightarrow P^{-1}p^j = e_j$

$\Rightarrow Ap^j = P\,\text{Diag}(x)\,P^{-1}p^j = P\,\text{Diag}(x)\,e_j = x_j P e_j = x_j p^j$

$\Rightarrow p^j$ is the eigenvector associated with the eigenvalue x_j

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ex: $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -\lambda \\ 1 & -1 \end{vmatrix}$$

$$= -\lambda\{-\lambda(2-\lambda)+1\} - \{2-\lambda-1\} + \{1-\lambda\}$$

$$= -\lambda\{-2\lambda + \lambda^2 + 1\}$$

$$= -\lambda\{\lambda^2 - 2\lambda + 1\} = -\lambda(\lambda-1)^2$$

$$\lambda = 0, 1, 1$$

$\lambda = 0$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 1$

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

check

$$P = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

137 $\quad P = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 2 \end{array}\right]$$

<u>check</u>

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$P \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} = A$$

(138)

Theorem: $A \in \mathbb{R}^{n\times n}$ is diagonalizable $\iff$ eigenvectors form a basis for $\mathbb{R}^n$

Pf: $(\Rightarrow)$ $A = P\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}P^{-1}$, set $P = [p^1\ p^2 \cdots p^n]$, $[P^{-1}]^T = [q^1\ q^2 \cdots q^n]$

$$\Rightarrow I_n = P^{-1}P = \begin{bmatrix}(q^1)^T \\ (q^2)^T \\ \vdots \\ (q^n)^T\end{bmatrix}[p^1\ p^2 \cdots p^n] = \begin{bmatrix}(q^1)^T p^1 & (q^1)^T p^2 & \cdots & (q^1)^T p^n \\ (q^2)^T p^1 & (q^2)^T p^2 & \cdots & (q^2)^T p^n \\ \vdots & & & \\ (q^n)^T p^1 & (q^n)^T p^2 & \cdots & (q^n)^T p^n\end{bmatrix}$$

$$\Rightarrow (q^i)^T p^j = \begin{cases} 0 & i \neq j \\ 1 & i = j\end{cases}$$

$\Rightarrow P^{-1}p^j = e_j$ the j^{th} unit coordinate vector $\Rightarrow Ap^j = P\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}P^{-1}p^j = P\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}e_j$

$= \lambda_j P e_j = \lambda_j p^j$

$\Rightarrow p^j$ is an eigenvector of A with eigenvalue λ_j.

$(\Leftarrow)$ Write $P = [p^1 \cdots p^n]$ $\Rightarrow P^{-1}AP = P^{-1}[Ap^1 \cdots Ap^n] = P^{-1}[\lambda_1 p^1\ \lambda_2 p^2 \cdots \lambda_n p^n]$

$= [\lambda_1 P^{-1}p^1 \cdots \lambda_n P^{-1}p^n] = P^{-1}P\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$

$\Rightarrow A = P\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}P^{-1}$