Ch. 5 Determinants : $|A| = \det(A)$

Definition: (i) $A \in \mathbb{R}^{1\times 1}$. Then $\det(A) = A$, ex. $\det(3) = 3$

(ii) $A \in \mathbb{R}^{2\times 2}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = ad - bc$

ex: $\det\left(\begin{bmatrix} 3 & 1 \\ -2 & -5 \end{bmatrix}\right) = (3)(-5) - (1)(-2) = -15 + 2 = -13$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ if } \det(A) \neq 0.$$

(iii) $A \in \mathbb{R}^{3\times 3}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \operatorname{Det}(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

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ex: $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 4 & 1 \\ 2 & 5 & 0 \end{bmatrix}$

$$|A| = 1 \cdot \begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 0 \\ 5 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = -5 - 0 + 4 = -1$$

$$|A| = 0 \cdot \begin{vmatrix} -1 & 4 \\ 2 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 0 - 1 + 0 = -1$$

$$|A| = -2 \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} + 4 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 4 - 5 = -1$$

(115) Cofactors of a Matrix

$A \in \mathbb{R}^{n \times n}$ the (i,j)-cofactor of A, $A(i,j)$ is the $(-1)^{i+j}$ times the determinant of of the matrix obtained from A by deleting the i^{th} row and j^{th} column from A.

$$A(i,j) = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2j} & \cdots & a_{2n} \\ \vdots & & & \vdots & & \\ a_{i1} & a_{i2} & & a_{ij} & \cdots & a_{in} \\ \vdots & & & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

<u>ex</u>: $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 8 & -7 & 2 \\ 1 & 0 & 2 & 1 \\ -1 & 4 & -1 & 2 \end{bmatrix}$, $A(2,3) = (-1)^{2+3} \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -1 & 4 & 2 \end{vmatrix}$

Brut force computation of the determinant of a 20×20 matrix requires about

$$20! = 2,432,902,008,176,640,000$$

multiplications!

Determinants are useful for theory and sometimes useful for computation. But we <u>must</u> develope other ways to compute them.

<u>Properties of Determinants</u>

(i) $\text{Det}(I) = 1$

(ii) if $A = \text{Diag}(x) = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & x_n \end{bmatrix}$, $\det(A) = x_1 x_2 \cdots x_n$

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Properties

(iii) $\det(A) = \det(A^T)$

(iv) A is upper triangular $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$

Then $\det(A) = a_{11} a_{22} \cdots a_{nn}$

(v) A is lower triangular. $\det(A) = a_{11} a_{22} a_{33} \cdots a_{nn}$

(vi) If a row or column of A is all zero, then $\det(A) = 0$

(vii) If B is obtained from A by interchanging any two rows (or columns)
Then $\det(B) = -\det(A)$.

119 (viii) If any two rows (columns) of A are the same, then $\det(A) = 0$.

(iv) $A, B \in \mathbb{R}^{n}$, $\det(AB) = \det(BA)$
$= \det(A)\det(B)$!

ex: (a) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ 7 & 2 & 1 & 0 \\ 13 & 37 & 11 & 4 \end{bmatrix}$, $\det(A) =$

(b) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -57 & 1 & 0 & 0 \\ 200 & 42 & 1 & 0 \\ 10^{10} & -20 & 8^{-37} & 1 \end{bmatrix}$, $\det(A) =$

(c) $A = \begin{bmatrix} 4 & -2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 2 & 10 \end{bmatrix}$, $\det(A) =$

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$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & & \vdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & & & A_{nn} \\ 0 & \cdots & & \end{bmatrix}$$

$A_{ii} \in \mathbb{R}^{k_i \times k_i}$, $A_{ij} \in \mathbb{R}^{k_i \times k_j}$

$$\det(A) = \det(A_{11}) \det(A_{22}) \cdots \det(A_{nn})$$

More Properties

(V) B is obtained from A by interchanging two rows
$\Rightarrow \det(B) = -\det(A)$

$A \in \mathbb{R}^{n \times n}$

(Vi) B is obtained from A by multiplying a row of A by α $\Rightarrow \det(B) = \alpha \det(A)$

$\det(\alpha A) = \alpha^n \det(A)$

(Vii) B is obtained from A by by adding a multiple of one row to another $\Rightarrow \det(B) = \det(A)$

(121) Find $\det \begin{bmatrix} -1 & 0 & -1 & 2 \\ -2 & 1 & 4 & -1 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix} = A$

(i) $\operatorname{Det}(GA) = \operatorname{Det}(G)\det(A)$ (ii) $G[A \mid I] = [GA \mid G]$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\ -2 & 1 & 4 & -1 & 0 & 1 & 0 & 0 \\ 5 & -1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 7 & -9 & -5 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$GA \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 9 & -8 & -3 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 0 & 1 \end{array}\right] G$$

$$\begin{aligned} \det(A) &= \det(G^{-1}GA) \\ &= \det(G^{-1})\det(GA) \\ &= 1 \cdot \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \det\begin{pmatrix} 9 & -8 \\ 3 & -1 \end{pmatrix} \\ &= -9 + 24 = 15 \end{aligned}$$

why does $\det(G^{-1}) = 1$?

$1 = \det(I) = \det(G^{-1}G) = \det(G^{-1})\det(G)$

$\det(G^{-1}) = \dfrac{1}{\det(G)} = [\det(G)]^{-1}$

(122) Find $\det \begin{bmatrix} 1 & -2 & 3 & -1 \\ 3 & -6 & 11 & 1 \\ -2 & 4 & -9 & 4 \\ 2 & -4 & 8 & 1 \end{bmatrix}$

(123) Is it ever the case that

$$\det(A+B) = \det(A) + \det(B)?$$

(a) $B = -A$

(b) Spps A is invertible $\Rightarrow$ $(A+B) = A(I + A^{-1}B)$

So when does $\det(A) + \det(B) = \det(A+B)$

$$= \det(A)\det(I + A^{-1}B)$$

or $1 + \det(A^{-1})\det(B) = \det(I + A^{-1}B)$

If $A^{-1}B$ is upper or lower triangular with zeros on the diagonal!

(c) Spps $B = \alpha A$ $\Rightarrow$ $(1+\alpha)\det(A) = \det(A + \alpha A)$
$A \in \mathbb{R}^{n \times n}$, $\alpha \neq 0$

$$= \det((1+\alpha)A)$$

$$= (1+\alpha)^n \det(A)$$

$\Rightarrow \det(A) = 0$ or

$(1+\alpha) = (1+\alpha)^n \iff$ $n=1$ or $\alpha = 0$

(124) Eigenvalues and Eigenvectors

<u>Definition</u>: Let $A \in \mathbb{R}^{n \times n}$. Then $x \in \mathbb{R}^n$, $x \neq 0$, is called an eigenvector of A if there is a scalar $\lambda \in \mathbb{C}$ such that $Ax = \lambda x$, where $\mathbb{C}^n$ is n-dimension complex space, $\mathbb{C}^n = \underbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}_{n \text{ times}}$

with $\mathbb{C} = \{x + iy : x, y \in \mathbb{R},\ i = \sqrt{-1}\}$

λ is called the eigenvalue of A associated with the eigenvector x.

<u>ex</u>: $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(125) <u>Ex</u>: (i) $A \in \mathbb{R}^{n\times n}$. $x \in \mathrm{Nul}(A)$

$$\Rightarrow \quad Ax = 0 = 0\cdot x$$

Therefore, x is an eigenvector of A associated with the eigenvalue $\lambda = 0$

(ii) $A \in \mathbb{R}^{n\times n}$, $\alpha \in \mathbb{R}$: Consider $B = A - \alpha I$

Then $x \in \mathrm{Nul}(B) \Leftrightarrow (A-\alpha I)x = 0$
$x \neq 0$

$$\Leftrightarrow \quad Ax - \alpha x = 0$$

$$\Leftrightarrow \quad Ax = \alpha x$$

$\Leftrightarrow$ x is an eigenvector of A with associated eigenvalue α

<u>Theorem</u>: λ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$ if and only if $\mathrm{Nul}(A - \lambda I) \neq \{0\}$.

(126) How to find eigenvectors

If λ is an eigenvalue for $A \in \mathbb{R}^{n \times n}$, then there is a vector x such that $Ax = \lambda x$

Therefore, $(A - \lambda I)x = 0$.

So we can find x by solving the homogeneous equation $[A - \lambda I \mid 0]$

ex: $A = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix}$,

$\lambda = 3$

$A - 3I = \begin{bmatrix} 1 & 4 & -2 \\ 1 & 1 & -1 \\ 3 & 6 & -4 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 1 & 1 & -1 & 0 \\ 3 & 6 & -4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 3 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$X = \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

Check:

$\begin{bmatrix} 4 & 4 & -2 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix}$

$= 3\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

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Thm: $A \in \mathbb{R}^{n \times n}$ has eigenvalue λ.

Then $\mathrm{Nul}(A - \lambda I)$ = set of all eigenvectors associated with λ.

Defn: If $A \in \mathbb{R}^{n \times n}$ has eigenvalue λ, we call $\mathrm{Nul}(A - \lambda I)$ the eigenspace of A associated with λ.

Thm: $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$

$$\mathrm{Nul}(A - \lambda I) \neq \{0\} \iff \boxed{\det(A - \lambda I) = 0}$$

$\iff$ λ is an eigenvalue of A

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<u>ex:</u> $A = \begin{bmatrix} 2 & 6 \\ 1 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 6 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6$$

$$= 2 - 3\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

$$\lambda = 4, -1$$

$A - 4I = \begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix}$ $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$A + I = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector associated with $\lambda = 4$

$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector associated with $\lambda = -1$

(129) $A = \begin{bmatrix} 0 & -3 & -1 \\ -1 & 2 & 1 \\ 3 & -9 & -4 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -3 & -1 \\ -1 & 2-\lambda & 1 \\ 3 & -9 & -4-\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} 2-\lambda & 1 \\ -9 & -(4+\lambda) \end{vmatrix} + \begin{vmatrix} -3 & -1 \\ -9 & -(4+\lambda) \end{vmatrix} + 3 \begin{vmatrix} -3 & -1 \\ 2-\lambda & 1 \end{vmatrix}$$

$= -\lambda[-(2-\lambda)(4+\lambda) + 9] + 3(4+\lambda) - 9 + 3[-3 + (2-\lambda)]$

$= \lambda(2-\lambda)(4+\lambda) - 9\lambda + \cancel{3\lambda} + \cancel{3} \cancel{-3} - 3\cancel{\lambda}$

$= \lambda(8 - 2\lambda - \lambda^2) - 9\lambda$

$= -\lambda^3 - 2\lambda^2 + 8\lambda - 9\lambda = -\lambda^3 - 2\lambda^2 - \lambda = -\lambda(\lambda^2 + 2\lambda + 1)$

$= -\lambda(\lambda+1)(\lambda+1)$

$\lambda = 0$

$$\begin{matrix} 0 & -3 & -1 \\ -1 & 2 & 1 \\ 3 & -9 & -4 \\ \hline 1 & -2 & -1 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \\ \hline 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{matrix} \quad \begin{pmatrix} 1/3 \\ \cancel{-1/3} \\ 1 \end{pmatrix} \quad x = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$\lambda = -1$

$$\begin{matrix} 1 & -3 & -1 \\ -1 & 3 & 1 \\ 3 & -9 & -3 \\ \hline 1 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad x = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Definition: Given $A \in \mathbb{R}^{n \times n}$ the characteristic polynomial for A is given by

$$p(\lambda) = \det(A - \lambda I).$$

$p(\lambda)$ is a polynomial of degree n.

Two step process for finding eigenvector eigenvalue pairs

(1) Solve the degree n polynomial equation
$$p(\lambda) = \det(A - \lambda I) = 0.$$
The roots of p are the eigenvalues of A.
There are n such roots counting multiplicity $\{\lambda_1, \lambda_2, \dots \lambda_n\}$

(2) Solve the homogeneous equation $(A - \lambda_i I)x = 0$ for the eigenvectors of A associated with the eigenvalue λ_i, $i = 1, \dots, n$.