Two nonstandard Bases

$$\mathcal{B}_1 = \{u^1, u^2, \dots, u^n\}, \quad \mathcal{B}_2 = \{v^1, v^2, \dots, v^n\}$$

$$\bar{U} = [u^1\ u^2 \dots u^n] \qquad \bar{V} = [v^1\ v^2 \dots v^n]$$

Given $y \in \mathbb{R}^n$, how do we get from $[y]_{\mathcal{B}_1}$ to $[y]_{\mathcal{B}_2}$?
That is how do we change the coordinate representation of y in $\mathcal{B}_1$ to the coordinates of y in $\mathcal{B}_2$?

Recall $[y]_{\mathcal{B}_1} = \bar{U}^{-1} y$ and $[y]_{\mathcal{B}_2} = \bar{V}^{-1} y$

so $\bar{V}^{-1}\bar{U}\,[y]_{\mathcal{B}_1} = \bar{V}^{-1}\bar{U}\,\bar{U}^{-1} y = \bar{V}^{-1} y = [y]_{\mathcal{B}_2}$

$$\begin{tikzcd}
\boxed{\mathcal{B}_1} \arrow[r, bend left, "\bar{U}"] \arrow[rr, bend left=50, "\bar{V}^{-1}\bar{U}"] & \boxed{\mathcal{S}} \arrow[l, bend left, "\bar{U}^{-1}"] \arrow[r, bend right, "\bar{V}^{-1}"'] & \boxed{\mathcal{B}_2} \arrow[l, bend right, "\bar{V}"'] \arrow[ll, bend left=50, "\bar{U}^{-1}\bar{V}"]
\end{tikzcd}$$

for standard basis

Change of basis matrix matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ is $\bar{V}^{-1}\bar{U}$

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ex: $\mathcal{B}_1 = \left\{ \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$ $\quad \mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \right\}$

What is the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$?

$$U = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & -1 \\ 4 & 3 & 2 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 7 \\ -1 & -1 & 0 \end{bmatrix} \Rightarrow \underbrace{V^{-1}U}_{\text{change of basis matrix}} [Y]_{\mathcal{B}_1} = [Y]_{\mathcal{B}_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 3 & 1 & 7 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -7 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$$

check

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 7 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -7 & 3 & 2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow V^{-1}U = \begin{bmatrix} 7 & -3 & -3 \\ -7 & 3 & 2 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & -1 \\ 4 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -19 & -4 & 4 \\ 15 & 1 & -6 \\ 6 & 2 & -1 \end{bmatrix}$$

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Change of basis between subspaces.

Thm Let $S \subset \mathbb{R}^n$ be a subspace and let

$\mathcal{B}_1 = \{u^1, \dots, u^k\}$, $\mathcal{B}_2 = \{v^1, \dots, v^k\}$ be two bases for S.

Suppose $[u^i]_{\mathcal{B}_2}$ are the coordinates of u^i in the basis $\mathcal{B}_2$, $i = 1..k$

and set $C = \big[[u^1]_{\mathcal{B}_2} \; [u^2]_{\mathcal{B}_2} \cdots [u^k]_{\mathcal{B}_2} \big] \in \mathbb{R}^{k \times k}$

Then for any $x \in S$, $[x]_{\mathcal{B}_2} = C\,[x]_{\mathcal{B}_1}$

Proof: Let $x \in S$ with $x = x_1 u^1 + x_2 u^2 + \dots + x_k u^k$
so that $[x]_{\mathcal{B}_1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$. Then

$$C[x]_{\mathcal{B}_1} = x_1 [u^1]_{\mathcal{B}_2} + x_2 [u^2]_{\mathcal{B}_2} + \dots + x_k [u^k]_{\mathcal{B}_2} = [x_1 u^1 + x_2 u^2 + \dots + x_k u^k]_{\mathcal{B}_2}$$
$$= [x]_{\mathcal{B}_2}$$

(11.0) ex: $\mathcal{B}_1 = \left\{ \underset{u^1}{\begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix}}, \underset{u^2}{\begin{pmatrix} 3 \\ -8 \\ 3 \end{pmatrix}} \right\}$, $\mathcal{B}_2 = \left\{ \underset{v^1}{\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}}, \underset{v^2}{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}} \right\}$

Does span $\mathcal{B}_1$ = span $\mathcal{B}_2$?

$$\begin{array}{l|rrr} v^{1T} & 1 & -3 & 2 \\ v^{2T} & -1 & 2 & 1 \\ u^{1T} & 1 & -5 & 8 \\ u^{2T} & 3 & -8 & 3 \\ \hline & 1 & -3 & 2 \\ & 0 & -1 & 3 \\ & 0 & -3 & 9 \\ & 0 & -2 & 6 \end{array}$$

Yes

What are $[u^1]_{\mathcal{B}_2}$ and $[u^2]_{\mathcal{B}_2}$?

$$\begin{array}{rr|rr} v^1 & v^2 & u^1 & u^2 \\ 1 & -1 & 1 & 3 \\ -3 & 2 & -5 & -8 \\ 2 & 1 & 8 & 3 \\ \hline 1 & -1 & 1 & 3 \\ 0 & -1 & -2 & 1 \\ 0 & 3 & 6 & -3 \\ \hline 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}$$

$\Rightarrow [u^1]_{\mathcal{B}_2} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $[u^2]_{\mathcal{B}_2} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Check

$$\begin{bmatrix} 1 & -1 \\ -3 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix} = u^1$$

$$\begin{bmatrix} 1 & -1 \\ -3 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ 3 \end{pmatrix} = u^1$$

If $[x]_{\mathcal{B}_1} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, what is x in $\mathbb{R}_3$?

$$\begin{bmatrix} 1 & 3 \\ -5 & -8 \\ 8 & 3 \end{bmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \\ 21 \end{pmatrix}$$

what is $[x]_{\mathcal{B}_2}$?

$$C = \left[[u^1]_{\mathcal{B}_2} \ [u^2]_{\mathcal{B}_2} \right] = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \Rightarrow [x]_{\mathcal{B}_2} = C[x]_{\mathcal{B}_1} = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

Check: $\begin{bmatrix} 1 & -1 \\ -3 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} 7 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \\ 21 \end{pmatrix}$ ✓

(111) Find the change of Basis matrix for

$$\mathcal{B}_1 = \left\{ \underset{u^1}{\begin{pmatrix} -10 \\ 4 \\ 2 \\ -5 \end{pmatrix}}, \underset{u^2}{\begin{pmatrix} -5 \\ 5 \\ 0 \\ -4 \end{pmatrix}}, \underset{u^3}{\begin{pmatrix} -8 \\ 5 \\ 2 \\ -6 \end{pmatrix}} \right\} \qquad \mathcal{B}_2 = \left\{ \underset{v^1}{\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}}, \underset{v^2}{\begin{pmatrix} -3 \\ 0 \\ 2 \\ -2 \end{pmatrix}}, \underset{v^3}{\begin{pmatrix} 6 \\ -3 \\ 2 \\ 0 \end{pmatrix}} \right\}$$

$$\begin{array}{ccc|ccc}
v^1 & v^2 & v^3 & u^1 & u^2 & u^3 \\
2 & -3 & 6 & -10 & -5 & -8 \\
1 & 0 & -3 & 4 & 5 & 5 \\
0 & 2 & 2 & 2 & 0 & 2 \\
-1 & -2 & 0 & -5 & -4 & -6 \\
\hline
1 & 0 & -3 & 4 & 5 & 5 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & -2 & -3 & -1 & 1 & -1 \\
0 & -3 & 12 & -18 & -15 & -18 \\
\hline
1 & 0 & -3 & 4 & 5 & 5 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & -1 & 1 & 1 & 1 \\
0 & 0 & 15 & -15 & -15 & -15 \\
\hline
1 & 0 & 0 & 1 & 2 & 2 \\
0 & 1 & 0 & 2 & 1 & 2 \\
0 & 0 & 1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$$\Rightarrow [u^1]_{\mathcal{B}_2} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad [u^2]_{\mathcal{B}_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad [u^3]_{\mathcal{B}_2} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$$

If $[x]_{\mathcal{B}_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

(a) what is $[x]_{\mathcal{B}_2}$?

$$x = \begin{array}{c} \begin{array}{ccc} u^1 & u^2 & u^3 \end{array} \\ \begin{bmatrix} -10 & -5 & -8 \\ 4 & 5 & 5 \\ 2 & 0 & 2 \\ -5 & -4 & -6 \end{bmatrix} \end{array} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -31 \\ 19 \\ -21 \end{pmatrix}$$

(b) what is $[x]_{\mathcal{B}_2}$?

$$C[x]_{\mathcal{B}_1} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ -4 \end{pmatrix}$$

Check:

$$\begin{bmatrix} 2 & -3 & 6 \\ 1 & 0 & -3 \\ 0 & 2 & 2 \\ -1 & -2 & 0 \end{bmatrix} \begin{pmatrix} 7 \\ 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 14 - 21 - 24 \\ 7 + 12 \\ -7 - 14 \end{pmatrix} = \begin{pmatrix} -31 \\ 19 \\ -21 \end{pmatrix}$$

112 Find a basis for Ran(A), Ran(A^T), and Nul(A).

$$A = \begin{bmatrix} 0 & 2 & -2 \\ 1 & -2 & 3 \\ -1 & 3 & 1 \\ -2 & 3 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} 0 & 2 & -2 \\ 1 & -1 & 3 \\ -1 & 3 & 1 \\ -2 & 3 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & -1 & 3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$$

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \\ 5 & 5 & -10 \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 1 & -2 \\ -2 & -2 & 4 \\ 5 & 5 & -10 \\ \hline 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$$

$[I, M]$

$\begin{bmatrix} -M \\ I_2 \end{bmatrix}$

$\begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Basis Ran(A) $= \left\{ \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \right\}$

Basis Nul(A) $= \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

Basis Ran(A^T) $= \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$

$$[I \quad M] \begin{bmatrix} -M \\ I \end{bmatrix} = -M + M = 0$$

$$[1 \quad 4] \begin{bmatrix} -4 \\ 1 \end{bmatrix} = -4 + 4 = 0$$

Ch. 5 Determinants : $|A| = \det(A)$

Definition: (i) $A \in \mathbb{R}^{1\times 1}$. Then $\det(A) = A$, ex. $\det(3) = 3$

(ii) $A \in \mathbb{R}^{2\times 2}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = ad - bc$

ex: $\det\left(\begin{bmatrix} 3 & 1 \\ -2 & -5 \end{bmatrix}\right) = (3)(-5) - (1)(-2) = -15 + 2 = -13$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ if } \det(A) \neq 0.$$

(iii) $A \in \mathbb{R}^{3\times 3}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathrm{Det}(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{22} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{32} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

11/3/05

~~Cofactor Expansion of the~~ Determinant

(The Laplace expansion of the determinant)

Cardano (1545) Italy 2×2, 3×3

Seki Takakazu (1683) Japan

Leibniz (1693) Germany

Cramer (1750) Cramer's Rule

Bezout (1779)

Vandermond (1779) functions on $\mathbb{R}^{n\times n}$ (multilinearity)

Lagrange (1772)

Gauss (1801) full generality and properties

matrices: Arthur Cayley $\boxed{1850's}$
John Sylvester

China 200 - 100 BCE lost by the beginning of CE

Cardano 1545
Seki 1683

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ex: $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 4 & 1 \\ 2 & 5 & 0 \end{bmatrix}$

$$|A| = 1 \cdot \begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 0 \\ 5 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = -5 - 0 + 4 = -1$$

$$|A| = 0 \cdot \begin{vmatrix} -1 & 4 \\ 2 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 0 - 1 + 0 = -1$$

$$|A| = -2 \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} + 4 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 4 - 5 = -1$$

115 Cofactors of a Matrix

$A \in \mathbb{R}^{n\times n}$ the (i,j)-cofactor of A, $A(i,j)$ is the $(-1)^{i+j}$ times the determinant of of the matrix obtained from A by deleting the i^{th} row and j^{th} column from A.

$$A(i,j) = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2j} & \cdots & a_{2n} \\ \vdots & & & \vdots & & \\ a_{i1} & a_{i2} & & a_{ij} & \cdots & a_{in} \\ \vdots & & & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

<u>ex</u>: $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 8 & -7 & 2 \\ 1 & 0 & 2 & 1 \\ -1 & 4 & -1 & 2 \end{bmatrix}$, $A(2,3) = (-1)^{2+3} \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -1 & 4 & -1 \end{vmatrix}$

(11.6) Cofactor Expansion (Laplace Expansion)

$A \in \mathbb{R}^{n\times n}$ $\quad i_0, j_0 \in \{1, \dots, n\}$ $\quad . \ n > 1$

$$|A| = det(A) = \sum_{i=1}^{n} a_{ij_0} A(i, j_0) = \sum_{j=1}^{n} a_{i_0 j} A(i_0, j)$$

j_0 column expansion $\qquad$ i_0 row expansion

ex: $A = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 3 & 1 & 1 & 0 \\ 5 & 1 & 2 & 1 \\ 4 & 3 & -3 & 2 \end{bmatrix}$

$$|A| = 3(-1)^{2+1} \begin{vmatrix} 1 & -1 & 2 \\ 1 & 2 & 1 \\ 3 & -3 & 2 \end{vmatrix} + (-1)^{2+2} \begin{vmatrix} 2 & -1 & 2 \\ 5 & 2 & 1 \\ 4 & -3 & 2 \end{vmatrix} + (-1)^{2+3} \begin{vmatrix} 2 & 1 & 2 \\ 5 & 1 & 1 \\ 4 & 3 & 2 \end{vmatrix}$$

$$= -3\left[(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} + (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ -3 & 2 \end{vmatrix} + 3(-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}\right] + \left[2(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} - 1(-1)^{1+2}\begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} 5 & 2 \\ 4 & -3 \end{vmatrix}\right]$$

$$- \left[5(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} + (-1)^{2+2}\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} + (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}\right]$$

$$= -3\left[(4+3) - (-2+6) + 3(-1-4)\right] + \left[2(4+3) + (10-4) + 2(-15-8)\right]$$
$$- \left[-5(2-6) + (4-8) - (6-4)\right]$$

$$= 36 - 26 - 14 = -4$$